Numerical Solutions of Non-linear Fractional Convection-Diffusion Equation Using Local Discontinuous Galerkin Method by B-spline Basic Functions

Hamid Safdari[a], Majid Rajabzadeh[a,*], Moein Khalighi[b]

[a]*Department of Applied Mathematics, Faculty of Mathematical sciences, Shahid Rajaee Teacher Training University, Tehran, Iran*
[b]*Department of computing, Faculty of Technology, University of Turku, Finland*

Abstract

In this paper, we solve the nonlinear fractional Convection - diffusion equation of order $\beta \in [1, 2]$ by the local discontinious Galerkin method by choices linear, quadratic and cubic basic B-Spline functions. These basic B-Spline functions Significantly reduces the error rate and significantly improve the accuracy and stability of the method . Selecting this type of basic functions greatly increases the accuracy of the method. We prove stability and optimal order of convergence $O(h^{k+1})$ for the fractional diffusion problem, Numerical results are provided to examine the accuracy of the proposed scheme and to compare it in different conditions.

Keywords:Basic function, stability, Order of convergence,

Introduction

Fractional calculations as a generalization of natural order calculations have been a major branch of mathematics in recent years. Fractional Differential Equations Generalized classical equations have many applications in physics, hydrology, financial mathematics, etc., including the fractional diffusion equation, which models the anomalous emission of a high-velocity particle. In recent decades, many researchers have analyzed and applied fractional differential equations with classical calculus in various sciences, including fluid mechanics, chemistry, engineering, and medicine [31, 32, 37]. As we know, differential equations and differential equation systems, both linear and nonlinear, are used in science and engineering to model and implement real-world observations, so how to model is very important. The models presented by partial equations (partial derivatives) are better than ordinary differential in terms of the application of some factors, and beyond them, random and fractional models are better than models derived from ordinary and partial differential equations and express the model in a better way.

We consider the fractional convection-diffusion equation

$$\frac{\partial C(y,t)}{\partial t} = \frac{\partial}{\partial x}\left(G\left(C\right)\frac{\partial C(y,t)}{\partial y}\right) + b\frac{\partial^{\beta} C(y,t)}{\partial y^{\beta}} + \rho(y,t), \quad (y,t) \in \mathbb{R} \times (0,T), \tag{1}$$
$$C(y,0) = C_0(y), \quad y \in \mathbb{R},$$

*Corresponding author

Email addresses: Hsafdari@sru.ac.ir (Hamid Safdari), majid.rajabzade@yahoo.com (Majid Rajabzadeh), moein.khalighi@utu.fi (Moein Khalighi)

in Equation (1), $b \geq 0$ is constant, $G(C(y,t)) \geq 0$ is bounded, $\frac{\partial}{\partial y}\left(G\left(C\right)\frac{\partial C(y,t)}{\partial y}\right)$, is nonlinear diffusion, $\beta \in [1,2]$ and Real-valued functions ρ, G are Lipschitz continuous. And the fraction derivative part is defined as follows:

$$\frac{\partial^{\beta} C}{\partial y^{\beta}} = \frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial y^2}\int_a^x (y-\eta)^{1-\beta} C(\eta,t) d\eta$$

In this paper we assume $\rho(0) = 0$, the discontinuous Galerkin method is a practical and important method for solving many differential equations. The accuracy of this method is very important depending on the choice of basic functions. In the last decade, Deng and Hesthaven [32], Xu and Hesthaven [28], Cifani [34] have solved the fractional diffusion equation by the Galerkin method using Legendre basic functions. In this paper, we solve this nonlinear equation by Garkin method using of basic B-Spline functions. B-spline, or base spline, is a spline function that has the least support in terms of a given degree of smoothness, and area separation. Any spline function of a certain degree can be written in terms of the linear combination of non-splines of the same degree. The main B-splines include points that are spaced apart. B-splines can be used to fit curves and numerically derive laboratory data.

This article is as follows. In section 1, we define some basic definitions and introduce the basic functions of B-Spline. In Section 2, We discretize the discontinuous Galerkin(LDG) method for solving the nonlinear diffusion equation. In sections 3 and 4, We show the method is stable and convergent. In Section 5 by solving three numerical examples, we solve the nonlinear diffusion equation with fractional derivatives by the LDG method and show the accuracy of the method.

1. Definitions and background

In this section, we give some basic definitions of fractional integral and fractional derivatives [37, 38]. Left and right Riemann-Liouville fractional integral of order α are defined as

$$ {}_aI_y^{\alpha}\, v(y) = \frac{1}{\Gamma(\alpha)}\int_a^y (y-\theta)^{\alpha-1} v(\theta) d\theta, \quad y > a, \quad \alpha \in \mathbb{R}^+, \tag{2}$$

$$ {}_yI_a^{\alpha}\, v(y) = \frac{1}{\Gamma(\alpha)}\int_y^a (\theta-y)^{\alpha-1} v(\theta) d\theta, \quad y < a, \quad \alpha \in \mathbb{R}^+, \tag{3}$$

where a is a real number. We define the Riemann-Liouville left and right fractional derivative of order α as follows:

$$ {}_{-\infty}D_y^{\alpha} v(y) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dy^n}\int_{-\infty}^y (y-\theta)^{n-\alpha-1} v(\theta) d\theta, \qquad \alpha \in [n-1,n)$$

$$ {}_yD_{\infty}^{\alpha} v(y) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dy}\right)^n\int_y^{\infty} (\theta-y)^{n-\alpha-1} v(\theta) d\theta \qquad \alpha \in [n-1,n).$$

suppose $C(y,t)$ has a solution of the equation (1) to the domain $[a,b]$. Assuming the $L+1$ discretization $a = y_1 < y_2 < y_3 < ... < y_{L+1} = b$, we provide a mesh by this notation $I_p = (y_p, y_{p+1}),\ h_p = y_{p+1} - y_p$. Now the piece wise polynomial space defined as follows:

$$\mathcal{S}^k = \left\{[a,b] \to \mathbb{R} \mid \gamma|_{I_p} \in \mathcal{P}^k(I_p),\ p = 1,2,3,...,L\right\}.$$

Suppose B-Spline polynomials are indicated by $\{W_{0,p}, W_{1,p}, ..., W_{k,p}\}$, where $W_{j,p} \in \mathcal{P}^j(I_p)$. For $k = 1$ we define basic functions

$$\xi_{1,p}(x) = \frac{1}{h}\begin{cases} y - y_p & [y_p, y_{p+1}] \\ y_{p+2} - y & [y_{p+1}, y_{p+2}] \\ 0 & otherwise \end{cases}$$

The quadratic B-splines $W_{2,p}$ which form a basis over the interval $[a, b]$ are define

$$\xi_{2,p}(x) = \frac{1}{h^2}\begin{cases} (y_{p+3} - y)^2 - 3(y_{p+2} - y)^2 + 3(y_{p+1} - y)^2, & [y_p, y_{p+1}], \\ (y_{p+3} - y)^2 - 3(y_{p+2} - y)^2, & [y_{p+1}, y_{p+2}], \\ (y_{p+3} - y)^2 & [y_{p+2}, y_{p+3}], \\ 0, & otherwise. \end{cases}$$

And for $k = 3$ the cubic B-splines $\xi_{3,p}$ at the knots are given by

$$\xi_{3,p}(x) = \frac{1}{h^3}\begin{cases} (y - y_p)^3, & [y_p, y_{p+1}], \\ (y - y_p)^3 - 4(y - y_{p+1})^3, & [y_{p+1}, y_{p+2}], \\ (y_{p+4} - y)^3 - 4(y_{p+3} - y)^3 & [y_{p+2}, y_{p+3}]\,, \\ (y_{p+4} - y)^3 & [y_{p+3}, y_{p+4}], \\ 0 & otherwise. \end{cases}$$

Where for $k = 1$, $\{W_{1,1}, W_{1,2}, ..., w_{1,L-1}, W_{1,L}\}$ forms a basis over the region $y \in [a, b]$. Each B-Spline covers two elements so that each element is covered by two B-splines. For $k = 2$, $\{W_{2,1}, W_{2,2}, W_{2,3}, ..., W_{2,L-1}, W_{2,L}\}$ forms a basis over the region $y \in [a, b]$. Each quadratic B-spline covers three elements so that each element is covered by three quadratic B-splines. And for $k = 3$, $\{W_{3,1}, W_{3,2}, W_{3,3}, W_{3,4}, ..., W_{3,L-1}, W_{3,L}\}$ forms a basis over the region $y \in [a, b]$. Each cubic B-spline covers four elements so that each element is covered by four cubic B-splines. Thus, we can convert each function in $\mathcal{P}^k(I_r)$ to a linear combination of these polynomials.

2. The LDG method

In the LDG method, we convert the initial equation into a system of several lower-order equations and solve this device with the Galerkin method. In fact, this technique (if stable) greatly increases the accuracy and improves the degree of convergence of the method.

For this problem we define three variables H, P, Q, and introduce

$$G\,(C)\,C_y = \left(\sqrt{G(C)}\right) A(C)_y$$

where $A(C)=\int^{C}\sqrt{G(C)}\mathrm{d}y$, which is placed in Equation (1) and becomes the following system:

$$C_t(y,t)-\left(\sqrt{b}H(y,t)+\sqrt{G(C)}Q(y,t)\right)_y=\rho(y,t) \tag{4}$$

$$H(y,t)-{}_aD_y^{\beta-2}P(y,t)=0 \tag{5}$$

$$P(y,t)-\sqrt{b}\frac{\partial G(y,t)}{\partial y}=0 \tag{6}$$

$$Q(y,t)-\frac{\partial A(C(y,t))}{\partial y}=0 \tag{7}$$

We seek (C,H,P,Q) as an approximation of $(C_h,H_h,P_h,Q_h)\in\mathcal{S}_h$ so that for any $\gamma,\mu,\sigma,\tau\in\mathcal{S}^k$, we have

$$\left(\frac{\partial C_h(y,t)}{\partial t},\gamma(y)\right)_{I_p}-\left(\left(\sqrt{b}H_h(y,t)+\sqrt{G(C_h)}Q_h(y,t)\right)_y,\frac{\partial\gamma(y)}{\partial y}\right)_{I_p}=(\rho(y,t),\gamma(y))_{I_p}, \tag{8}$$

$$(H_h(y,t),\mu(y))_{I_p}-\left({}_aD_y^{\beta-2}P_h(y,t),\mu(y)\right)_{I_p}=0, \tag{9}$$

$$(P_h(y,t),\sigma(y))_{I_p}-\left(\sqrt{b}\frac{\partial C_h(y,t)}{\partial y},\frac{\partial\sigma(y)}{\partial y}\right)_{I_p}=0, \tag{10}$$

$$(Q_h(y,t),\tau(y))_{I_p}-\left(\frac{\partial A(C_h(y,t))}{\partial y},\frac{\partial\tau(y)}{\partial y}\right)_{I_p}=0, \tag{11}$$

$$(C_h(y,0),\gamma(y))_{I_p}=(C_0(y),\gamma(y))_{I_p}. \tag{12}$$

Note that $(C,\gamma)_I=\int_I C(y)\gamma(y)\mathrm{d}y$ Is defined. Now considering that

$$C^{\pm}(\lambda_j)=\lim_{\lambda\to\lambda_j^{\pm}}C(y),\quad \bar{C}=\frac{C^{+}+C^{-}}{2},\quad [\![C]\!]=C^{+}-C^{-},$$

We define numerical flux as follows

$$\hat{C}=\mathcal{C}(C^{-},C^{+}),\qquad \hat{H}=\hat{\mathcal{H}}(C_h^{-},C_h^{+}),\qquad \hat{P}=\mathcal{P}(P^{-},P^{+}).$$

For the high order derivative part, we can define

$$\hat{C}_{p+1}=C_{p+1}^{-},\quad \hat{P}_{p+1}=P_{p+1}^{+}\quad \hat{H}_{p+1}=H_{p+1}^{+},\quad p=0,1,2,...,L-1,$$

or

$$\hat{C}_{p}=C_{p+1}^{+},\quad \hat{P}_{p}=P_{p+1}^{-}\quad \hat{H}_{p+1}=H_{p+1}^{+},,\quad p=1,2,...,L$$

At the boundary, we defined the flux as follwos

$$C_{L+1}=C(b,t),\quad \hat{P}_{L+1}=P_{L+1}^{-}+\frac{\alpha}{h}[C_{L+1}]$$

or

$$C_1=C(b,t),\quad \hat{P}_1=P_1^{-}+\frac{\alpha}{h}[C_1]$$

Where α is a positive constant. let us also introduce the flux

$$\sqrt{\hat{G}(C_p)}=\sqrt{\hat{G}\left(C(y_p^{-}),...,\frac{\partial^p C(y_r^{-})}{\partial y},C(y_r^{+}),...,\frac{\partial^p C(y_p^{+})}{\partial y}\right)}$$

$$= \lambda_0 \frac{[\![A(C_p)]\!]}{\Delta y} + \overline{A(C_p)_y} + \sum_{m=1}^{\lfloor k/2 \rfloor} \lambda_m \Delta y^{2m-1} [\![\partial_y^m A(C_p)]\!] ,$$

Where $A(C) = \int^C \sqrt{G} dy$ and the weights $\{\lambda_0, ..., \lambda_{\lfloor k/2 \rfloor}\}$ fulfill the following admissibility condition: there exist $\epsilon \in (0,1)$ and $\omega \geqslant 0$ such that

$$\sum_{p\in\mathbb{Z}} \sqrt{\hat{G}(C_p)} [\![C_p]\!] \geqslant \omega \sum_{p\in\mathbb{Z}} \frac{[\![A(C_p)]\!]}{\Delta y} [\![C_p]\!] - \epsilon \sum_{p\in\mathbb{Z}} \int_{I_p} G(C)(C_y)^2 \, dt. \tag{13}$$

Using integration by parts to (8) and by putting fluxes obtained:

$$\left(\frac{\partial C_h}{\partial t}, \gamma\right)_{I_p} + \left(\left(\sqrt{b} H_h + \sqrt{G(C_h)} Q_h\right), \frac{\partial \gamma}{\partial y}\right)_{I_p} - \sqrt{b} \hat{H}_h \gamma |_{y_p^+}^{y_{p+1}^-} - \sqrt{\hat{G}(C_h)} \hat{Q}_h \gamma |_{y_p^+}^{y_{p+1}^-} - (\rho, \gamma)_{I_p} = 0, \tag{14}$$

$$(H_h(y,t), \mu(y))_{I_p} - \left({}_a D_y^{\beta-2} P_h(y,t), \mu(y)\right)_{I_p} = 0, \tag{15}$$

$$(P_h(y,t), \sigma(y))_{I_p} + \left(\sqrt{b} C_h(y,t), \frac{\partial \sigma(y)}{\partial y}\right)_{I_p} - \sqrt{b} \hat{C}_h(y,t) \sigma(y) |_{y_p^+}^{y_{p+1}^-} = 0, \tag{16}$$

$$(Q_h(y,t), \tau(y))_{I_p} + \left(A(C_h), \frac{\partial \tau(y)}{\partial y}\right)_{I_p} - \hat{A}(C_h) \tau(y) |_{y_p^+}^{y_{p+1}^-} = 0, \tag{17}$$

$$(C_h(y,0), (y))_{I_p} = (C_0(y), \gamma(y))_{I_p}. \tag{18}$$

Our goal is to finding $\tilde{\mathbf{R}} = \left(\tilde{C}, \tilde{H}, \tilde{P}, \tilde{Q}\right)^T$ such that

$$\tilde{C}(y,t) = \sum_{p=1}^{L} \sum_{z=1}^{M} U_{z,p}(t) W_{z,p}(y), \quad \tilde{H}(y,t) = \sum_{p=1}^{L} \sum_{z=1}^{M} A_{z,p}(t) W_{z,p}(y),$$

$$\tilde{P}(y,t) = \sum_{p=1}^{L} \sum_{z=1}^{M} P_{z,p}(t) W_{z,p}(y), \quad \tilde{Q}(y,t) = \sum_{p=1}^{L} \sum_{z=1}^{k} E_{z,p}(t) W_{z,p}(y),$$

where they are functions satisfying (14)-(18) for all $C, \gamma, \mu, \tau \in \mathcal{P}^k(I_r)$, $p \in \{1, 2, ..., L\}$.

3. Stability

We now discuss the stability of the LDG method. To do this, we first define:

$$\mathscr{M}(C,H,P,Q,\gamma,\mu,\sigma,\tau) = \int_0^T \sum_{p=1}^{L} (C_t, \gamma)_{I_p} \mathrm{d}t + \int_0^T \sum_{p=1}^{L} \left(\left(\sqrt{b} H + \sqrt{G(C)} Q\right), \frac{\partial \gamma}{\partial y}\right)_{I_p} \mathrm{d}t$$

$$- \int_0^T \sum_{p=1}^{L-1} \left(\sqrt{b} \hat{H} \gamma + \sqrt{\hat{G}(C_h)} \hat{Q} \gamma\right) |_{y_p^+}^{y_{p+1}^-} \mathrm{d}t - \int_0^T \sum_{p=1}^{L} (\rho, \gamma)_{I_p} \mathrm{d}t$$

$$+ \int_0^T \sum_{p=1}^{L} (H(y,t), \mu(y))_{I_p} \mathrm{d}t - \int_0^T \sum_{p=1}^{L} \left({}_a D_y^{\beta-2} P(y,t), \mu(y)\right)_{I_p} \mathrm{d}t$$

$$+ \int_0^T \sum_{p=1}^{L} (P(y,t), \sigma(y))_{I_p} \mathrm{d}t + \int_0^T \sum_{p=1}^{L} \left(\sqrt{b} C(y,t), \frac{\partial \sigma(y)}{\partial y}\right)_{I_p} \mathrm{d}t$$

$$-\int_0^T \sum_{p=1}^{L-1} \sqrt{b}\hat{C}(y,t)\sigma(y)|_{y_p^+}^{y_{p+1}^-} \mathrm{d}t + \int_0^T \sum_{p=1}^{L} (Q(y,t),\tau(y))_{I_p} \mathrm{d}t$$

$$+\int_0^T \sum_{p=1}^{L} \left(A(C), \frac{\partial \tau(y)}{\partial y}\right)_{I_p} \mathrm{d}t - \int_0^T \sum_{p=1}^{L-1} \hat{A}(C)\tau(y)|_{y_p^+}^{y_{p+1}^-} \mathrm{d}t$$

Notice $\mathscr{M}(C,H,P,Q,\gamma,\mu,\sigma,\tau) = 0$ for any (γ,μ,σ,τ) if (C,H,P,Q) is a solution. Assuming fluxes

$$\hat{P}_{p+1} = P_{p+1}^+, \quad \hat{A}(C)_{p+1} = A(C_{p+1}^+), \quad 1 \le p \le L-1,$$

for the boundary condition, we define

$$\hat{C}_{L+1} = C(b,t), \qquad \hat{P}_{L+1} = P_{L+1}^- + \frac{\beta}{h}[C_{L+1}].$$

We will now have:

$$\begin{aligned}
\mathscr{M}(C,H,P,Q,\gamma,\mu,\sigma,\tau) = &\int_0^T \sum_{p=1}^{L} (C_t,\gamma)_{I_p} \mathrm{d}t + \int_0^T \sum_{p=1}^{L} \left(\sqrt{b}H, \gamma_y\right) \mathrm{d}t + \int_0^T \sum_{p=1}^{L} \left(\sqrt{G(C)}Q, \gamma_y\right)_{I_p} \mathrm{d}t \\
&- \int_0^T \sum_{p=1}^{L} (\rho,\gamma)_{I_p} \mathrm{d}t + \int_0^T \sum_{p=1}^{L} (H(y,t),\mu(y))_{I_p} \mathrm{d}t \\
&- \int_0^T \sum_{p=1}^{L} \left({}_aD_y^{\beta-2}P(y,t),\mu(y)\right)_{I_p} \mathrm{d}t + \int_0^T \sum_{p=1}^{L} (P(y,t),\sigma(y))_{I_p} \mathrm{d}t \\
&+ \int_0^T \sum_{p=1}^{L} \left(\sqrt{b}C(y,t), \frac{\partial \sigma(y)}{\partial y}\right)_{I_p} \mathrm{d}t + \int_0^T \sum_{p=1}^{L} (Q(y,t),\tau(y))_{I_p} \mathrm{d}t \\
&+ \int_0^T \sum_{p=1}^{L} \left(A(C), \frac{\partial \tau(y)}{\partial y}\right)_{I_p} \mathrm{d}t - \int_0^T \sum_{p=1}^{L-1} \sqrt{b}\hat{H}_{p+1} [\![\gamma]\!]_{p+1}\, dt \\
&- \int_0^T \sum_{p=1}^{L-1} (\sqrt{\hat{G}(C)}\hat{Q})_{p+1} [\![\gamma]\!]_{p+1} dt - \int_0^T \sum_{p=1}^{L-1} \sqrt{b}\hat{C}_{p+1} [\![\sigma]\!]_{p+1} \mathrm{d}t \\
&- \int_0^T \sum_{p=1}^{L-1} (\hat{A}(C))_{p+1} [\![\tau]\!]_{p+1} \mathrm{d}t - \int_0^T \sqrt{b}\left(\hat{H}_1\gamma_1^+ - \hat{H}_{L+1}\gamma_{L+1}^-\right) \mathrm{d}t \\
&- \int_0^T \left(\sqrt{\hat{G}_1(C)}\hat{Q}_1\gamma_1^+ - \sqrt{\hat{G}_{L+1}(C)}\hat{Q}_{L+1}\gamma_{L+1}^-\right) \mathrm{d}t \\
&- \int_0^T \sqrt{b}\left(\hat{C}_1\sigma_1^+ - \hat{C}_{L+1}\sigma_{L+1}^-\right) \mathrm{d}t - \int_0^T \left(\hat{A}_1(C)\tau_1^+ - \hat{A}_{L+1}(C)\tau_{L+1}^-\right) \mathrm{d}t
\end{aligned} \tag{20}$$

Lemma 1. *By setting $(\gamma,\mu,\sigma,\tau)=(C,H,-Q,P)$ in (20), we defining we achieve the following result*

$$\mathscr{M}(C,H,P,Q,C,H,-Q,P)=\|C(y,T)\|^2-\|C_0\|^2+\int_0^T\sum_{p=1}^{L}(H,H)_{I_p}\mathrm{d}t$$

$$-\int_0^T\sum_{p=1}^{L}(\rho(y,t),C(y,t))_{I_p}-\int_0^T\sum_{p=1}^{L}\left({}_aD_y^{\beta-2}P(y,t),H\right)_{I_p}\mathrm{d}t+\int_0^T\frac{\sqrt{b}}{h}\beta\left(C_{L+1}^{-}\right)^2\mathrm{d}t$$

Proof. If we suppose $(\gamma,\mu,\sigma,\tau)=(C,H,-Q,P)$ in (20), and using integration by parts formula

$$(Q,C_y)_{I_p}+(Q_y,C)_{I_p}=QC|_{y_p^+}^{y_{p+1}^-},$$
$$(A(C),P_y)_{I_p}+(A(C)_y,P)_{I_p}=A(C)P|_{y_p^+}^{y_{p+1}^-},$$

we have

$$\sum_{p=1}^{L}\left(\sqrt{G(C)}Q,C_y\right)_{I_p}+\sum_{p=1}^{L}\left(\sqrt{b}C,Q_y\right)_{I_p}+\sum_{p=1}^{L-1}\left(\sqrt{\hat{G}}\hat{Q}\right)_{p+1}[\![C]\!]_{p+1}+\sum_{p=1}^{L-1}\left(\sqrt{b}\hat{C}\right)_{p+1}[\![Q]\!]_{p+1}=\sqrt{b}C_1^{+}Q_1^{+}-\sqrt{b}C_{L+1}^{-}Q_{L+1}^{-},$$

$$\sum_{p=1}^{L}\left(\sqrt{b}H,C_y\right)_{I_p}+\sum_{p=1}^{L}\left(\sqrt{b}H_y,C\right)_{I_p}+\sum_{p=1}^{L-1}\left(\sqrt{b}\hat{H}\right)_{p+1}[\![C]\!]_{p+1}+\sqrt{b}\sum_{p=1}^{L-1}\left(\hat{C}\right)_{p+1}[\![H]\!]_{p+1}=\sqrt{b}H_1^{+}C_1^{+}-\sqrt{b}H_{L+1}^{-}C_{L+1}^{-}.$$

Then

$$\mathscr{M}(C,H,P,Q,C,H,-Q,P)=\int_0^T\sum_{p=1}^{L}(C_t,C)_{I_p}dt-\int_0^T\sum_{p=1}^{L}(\rho,C)_{I_p}\mathrm{d}t$$

$$+\int_0^T\sum_{p=1}^{L}(H,H)_{I_p}dt-\int_0^T\sum_{p=1}^{L}\left({}_aD_y^{\beta-2}P,{}_aD_y^{\beta-2}P\right)_{I_p}\mathrm{d}t$$

$$+\int_0^T\frac{\sqrt{b}}{h}\varepsilon\left(C_{L+1}^{-}\right)^2\mathrm{d}t. \tag{21}$$

The proof is complete. □

Theorem 2. *The semi-discrete scheme* (8)-(12) *is stable, and $\forall T>0$ we have $\|u_h(y,T)\|\le\|u_0(y)\|$.*

Proof. By considering Galerkin orthogonality, $\mathscr{M}(C_h,H_h,P_h,Q_h,C_h,H_h,-Q_h,P_h)=0$, Lemma 1 yields

$$\mathscr{M}(C,H,P,Q,C,H,-Q,P)=\|C(y,T)\|^2-\|C_0\|^2+\int_0^T\sum_{p=1}^{L}(H,H)_{I_p}\mathrm{d}t$$

$$-\int_0^T\sum_{p=1}^{L}(\rho(y,t),C(y,t))_{I_p}-\int_0^T\sum_{p=1}^{L}\left({}_aD_y^{\beta-2}P(y,t),H\right)_{I_p}\mathrm{d}t+\int_0^T\frac{\sqrt{b}}{h}\beta\left(C_{L+1}^{-}\right)^2\mathrm{d}t$$

Finally, the boundary condition leads to $\|C_h(y,T)\|\le\|C_0(y)\|$ and completes the proof. □

4. Error estemate

In this section we assume $\rho = 0$, $G \equiv 1$ and $A(C) = C$. For fractional diffusion, (8)-(12) we have

$$((C_h)_t, \gamma)_{I_p} + (H_h, \gamma_y)_{I_p} - \hat{H}_h \gamma|_{y_p^+}^{y_{p+1}^-} + (Q_h, \gamma_y)_{I_p} - (\hat{Q}_h \gamma)|_{y_p^+}^{y_{p+1}^-} = 0, \tag{22}$$

$$(H_h(y,t), \mu(y))_{I_p} - \left({}_a D_y^{\beta-2} P_h(y,t), \mu(y)\right)_{I_p} = 0, \tag{23}$$

$$(P_h(y,t), \sigma(y))_{I_p} + \left(\sqrt{b} C_h(y,t), \frac{\partial \sigma(y)}{\partial y}\right)_{I_p} - \sqrt{b}\hat{C}_h(y,t)\sigma(y)|_{y_p^+}^{y_{p+1}^-} = 0, \tag{24}$$

$$(Q_h(y,t), \tau(y))_{I_p} + \left(C_h, \frac{\partial \tau(y)}{\partial y}\right)_{I_p} - \hat{C}_h \tau(y)|_{y_p^+}^{y_{p+1}^-} = 0, \tag{25}$$

$$(C_h(y,0), \gamma(y))_{I_p} = (C_0(y), \gamma(y))_{I_p}. \tag{26}$$

now, the scheme can be written as follows:

$$\begin{aligned}
\mathscr{M}(C,H,P,Q,\gamma,\mu,\sigma,\tau) = {} & \int_0^T \sum_{p=1}^{L} (C_t, \gamma)_{I_p} \mathrm{d}t + \int_0^T \sum_{p=1}^{L} (H, \gamma_y)_{I_p} \mathrm{d}t \\
& + \int_0^T \sum_{p=1}^{L} (H(y,t), \mu(y))_{I_p} \mathrm{d}t - \int_0^T \sum_{p=1}^{L} \left({}_a D_y^{\beta-2} P(y,t), \mu(y)\right)_{I_p} \mathrm{d}t + \int_0^T \sum_{p=1}^{L} (P(y,t), \sigma(y))_{I_p} \mathrm{d}t \\
& + \int_0^T \sum_{p=1}^{L} \left(\sqrt{b} C(y,t), \frac{\partial \sigma(y)}{\partial y}\right)_{I_p} \mathrm{d}t + \int_0^T \sum_{p=1}^{L} (Q(y,t), \tau(y))_{I_p} \mathrm{d}t + \int_0^T \sum_{p=1}^{L} \left(C(y,t), \frac{\partial \tau(y)}{\partial y}\right)_{I_p} \mathrm{d}t \\
& + \int_0^T \sum_{p=1}^{L} (Q(y,t), \gamma_y)_{I_p} \mathrm{d}t + \int_0^T \sum_{p=1}^{L-1} H_{p+1}^+ [\![\gamma]\!]_{p+1} \mathrm{d}t + \int_0^T \sum_{p=1}^{L-1} \sqrt{b} C_{p+1}^- [\![\sigma]\!]_{p+1} \mathrm{d}t \\
& + \int_0^T \sum_{p=1}^{L-1} C_{p+1}^+ [\![\tau]\!]_{p+1} \mathrm{d}t + \int_0^T \sum_{p=1}^{L-1} Q_{p+1}^+ [\![\gamma]\!]_{p+1} \mathrm{d}t + \int_0^T H_1^+ \gamma_1^+ \mathrm{d}t \\
& - \frac{\sqrt{b}\beta}{h} \int_0^T C_{L+1}^- \sigma_{L+1}^- \mathrm{d}t + \int_0^T C_1^+ \sigma_1^- \mathrm{d}t + \int_0^T Q_1^+ \gamma_1^+ \mathrm{d}t
\end{aligned} \tag{27}$$

Let $\mathscr{P}^{\pm} e$ be the L^2-projection of$e$ into $\mathscr{V}^k$, i.e, $\mathscr{P}^{\pm} e$ is the $\mathscr{V}^k \cap L^2(R)$ function satisfying

$$\int_{I_p} \left(\mathscr{P}^{\pm} e(y) - e(y)\right) \zeta_{jp}(y) \mathrm{d}y = 0. \tag{28}$$

Where $j = 1, 2, .., L,\ \ p = \{0, 1, 2, ..., k-1\}$, and $\mathscr{P}^{\pm} C_{p+1} = C(y_{p+1}^{\pm})$. Note that $e_C = C - C_h$, $e_H = H - H_h$, $e_P = P - P_h$, and $e_Q = Q - Q_h$, then $\mathscr{P}^- e_C = \mathscr{P}^- C - C_h$, $\mathscr{P}^+ e_P = \mathscr{P}^+ P - P_h$, $\mathscr{P}^+ e_H = \mathscr{P}^+ H - H_h$, and $\mathscr{P} e_Q = \mathscr{P} Q - Q_h$ for all $(\gamma, \mu, \sigma, \tau) \in (\Omega, \mathcal{T}) \times (\Omega, \mathcal{T}) \times (\Omega, \mathcal{T}) \times (\Omega, \mathcal{T})$,

$$\mathscr{M}(C, H, P, Q; \gamma, \mu, \sigma, \tau) = \mathscr{P}(\gamma, \mu, \sigma, \tau). \tag{29}$$

Hence, $\mathscr{M}\left(e_C, e_H, e_P, e_Q; \gamma, \mu, \sigma, \tau\right) = 0$ and we gain

$$\mathscr{M}\left(\mathscr{P}^- e_C, \mathscr{P}^+ e_H, \mathscr{P}^+ e_P, \mathscr{P} e_Q; \mathscr{P}^- e_C, \mathscr{P}^+ e_H, -\mathscr{P} e_Q, \mathscr{P}^+ e_P\right)$$

$$= \mathscr{M}\left(\mathscr{P}^- e_C - e_C, \mathscr{P}^+ e_H - e_H, \mathscr{P}^+ e_P - e_P, \mathscr{P} e_Q - e_Q; \mathscr{P}^- e_C, \mathscr{P}^+ e_H, -\mathscr{P} e_Q, \mathscr{P}^+ e_P\right)$$

$$= \mathscr{M}\left(\mathscr{P}^- C - C, \mathscr{P}^+ H - \mathrm{H}, \mathscr{P}^+ P - P, \mathscr{P} Q - Q; \mathscr{P}^- e_C, \mathscr{P}^+ e_H, -\mathscr{P} e_Q, \mathscr{P}^+ e_P\right).$$

Substitute $(\mathscr{P}^- C - C, \mathscr{P}^+ H - \mathrm{H}, \mathscr{P}^+ P - P, \mathscr{P} Q - Q; \mathscr{P}^- e_C, \mathscr{P}^+ e_H, -\mathscr{P} e_Q, \mathscr{P}^+ e_P)$ into (27) proves the Lemma.

Lemma 3. *Form* (27) *is transformed by changing the given variables as follows:*

$$\begin{aligned}
&\mathscr{M}\left(\mathscr{P}^- C - C, \mathscr{P}^+ H - \mathrm{H}, \mathscr{P}^+ P - P, \mathscr{P} Q - Q; \mathscr{P}^- e_C, \mathscr{P}^+ e_H, -\mathscr{P} e_Q, \mathscr{P}^+ e_P\right) \\
&\qquad \leq \int_0^T \sum_{p=1}^{L} \left((\mathscr{P}^- C)_t - C_t, \mathscr{P}^- e_C\right)_{I_p} \mathrm{d}t + c_{T,a,b} h^{2k+2} + \frac{1}{c_{T,a,b}} \int_0^T \sum_{p=1}^{L} \|\mathscr{P} e_H\|_{I_p}^2 \,\mathrm{d}t \\
&\qquad + \int_0^T \frac{\sqrt{b}\beta}{h} |(\mathscr{P}^- e_C)_{L+1}|^2 \mathrm{d}t + \int_0^T \sum_{p=1}^{L} \left\|\mathscr{P}^+ e_H\right\|_{I_p}^2 \mathrm{d}t,
\end{aligned}$$

Proof. From (27) we have

$$\begin{aligned}
&\mathscr{M}\left(\mathscr{P}^- C - C, \mathscr{P}^+ H - \mathrm{H}, \mathscr{P}^+ P - P, \mathscr{P} Q - Q; \mathscr{P}^- e_C, \mathscr{P}^+ e_P, -\mathscr{P} e_Q, \mathscr{P}^+ e_P\right) \\
&= \int_0^T \sum_{p=1}^{L} \left((\mathscr{P}^- C)_t - C_t, \mathscr{P}^- e_C\right)_{I_p} \mathrm{d}t + \int_0^T \sum_{p=1}^{L} \Big((\mathscr{P}^+ H - H), (\mathscr{P}^- e_C)_y\Big)_{I_p} \mathrm{d}t + \int_0^T \sum_{p=1}^{L} \left((\mathscr{P}^+ H - H), \mathscr{P}^+ e_H\right)_{I_p} \mathrm{d}t \\
&\quad - \int_0^T \sum_{p=1}^{L} \Big({}_a D_y^{\beta-2}(\mathscr{P}^+ P - P), \mathscr{P}^+ e_H\Big)_{I_p} \mathrm{d}t - \int_0^T \sum_{p=1}^{L} \left((\mathscr{P}^+ P - P, \mathscr{P} e_Q\right)_{I_p} \mathrm{d}t + b \int_0^T \sum_{p=1}^{L} \left(\mathscr{P}^- C - C), (\mathscr{P} e_Q)_y\right)_{I_p} \mathrm{d}t \\
&\quad + \int_0^T \sum_{p=1}^{L} \left((\mathscr{P} Q - Q), \mathscr{P}^+ e_P\right)_{I_p} \mathrm{d}t + \int_0^T \sum_{p=1}^{L} \Big(\mathscr{P}^- C - C, (\mathscr{P}^+ e_P)_y\Big)_{I_p} \mathrm{d}t + \int_0^T \sum_{p=1}^{L} \Big(\mathscr{P} Q - Q, (\mathscr{P}^- e_C)_y\Big)_{I_p} \mathrm{d}t \\
&\quad + \int_0^T \sum_{p=1}^{L-1} (\mathscr{P}^+ H - H)_{p+1}^+ [\![\mathscr{P}^- e_C]\!]_{p+1} \mathrm{d}t - \sqrt{b} \int_0^T \sum_{p=1}^{L-1} (\mathscr{P}^- C - C)_{p+1}^- [\![\mathscr{P} e_Q]\!]_{p+1} \mathrm{d}t \\
&\quad + \int_0^T \sum_{p=1}^{L-1} (\mathscr{P}^- C - C)_{p+1}^+ [\![\mathscr{P}^+ e_P]\!]_{p+1} \mathrm{d}t + \int_0^T \sum_{p=1}^{L-1} (\mathscr{P} Q - Q)_{p+1}^+ [\![\mathscr{P}^- e_C]\!]_{p+1} \mathrm{d}t \\
&\quad + \int_0^T (\mathscr{P}^+ H - H)_1^+ [\![\mathscr{P}^- e_C]\!]_1^+ \mathrm{d}t + \frac{\sqrt{b}\beta}{h} \int_0^T (\mathscr{P}^- C - C)_{L+1}^+ [\![\mathscr{P}^- e_Q]\!]_{L+1}^+ \mathrm{d}t \\
&\quad - \int_0^T (\mathscr{P}^- C - C)_1^+ [\![\mathscr{P} e_Q]\!]_1^- \mathrm{d}t + \int_0^T (\mathscr{P} Q - Q)_1^+ [\![\mathscr{P}^- e_C]\!]_1^- \mathrm{d}t.
\end{aligned}$$

Since $(\mathscr{P} e_Q)_y \in \mathcal{P}^{k-1}$, $(\mathscr{P}^- e_C)_y \in \mathcal{P}^{k-1}$, $\mathscr{P}^+ e_H \in \mathcal{P}^{k-1}$, $\mathscr{P} e_Q \in \mathcal{P}^k$, by considering the properties of the projection $\mathscr{P}^\pm$ and $\mathscr{P}$: $\Big(\mathscr{P}^+ H - H, (\mathscr{P}^- e_C)_y\Big)_{I_p} = 0$,
$(\mathscr{P}^+ P - P, \mathscr{P} e_Q)_{I_p} = 0$, $\Big((\mathscr{P}^- C - C, (\mathscr{P}^+ e_Q)_y\Big)_{I_p} = 0$, $(\mathscr{P} Q - Q, \mathscr{P}^+ e_P)_{I_p} = 0$,
$(\mathscr{P}^- C - C, (\mathscr{P}^+ e_P)_y)_{I_r} = 0, (\mathscr{P}^- H - H)_{p+1} = 0$, and $(\mathscr{P}^- C - C)_{p+1} = 0$, therefore

$$\mathscr{M}\left(\mathscr{P}^- C - C, \mathscr{P}^+ H - \mathrm{H}, \mathscr{P}^+ P - P, \mathscr{P} Q - Q; \mathscr{P}^- e_C, \mathscr{P}^+ e_H, -\mathscr{P} e_Q, \mathscr{P}^+ e_P\right)$$

$$
\begin{aligned}
&= \int_0^T \sum_{p=1}^{L} \left((\mathscr{P}^- C)_t - C_t, \mathscr{P}^- e_C\right)_{I_p} \mathrm{d}t + \int_0^T \sum_{p=1}^{L} \left(\mathscr{P}^+ H - H, \mathscr{P}^+ e_H\right)_{I_r} \mathrm{d}t \\
&\quad - \int_0^T \sum_{p=1}^{L} \left({}_aD_y^{\beta-2}(\mathscr{P}^+ P - P), \mathscr{P}^+ e_H\right)_{I_p} \mathrm{d}t - \sqrt{b}\int_0^T (\mathscr{P}^+ H - H^-)_{L+1}[\mathscr{P}^- e_C]_{L+1}^- \mathrm{d}t.
\end{aligned}
$$

Using the projection feature and lemma 2 in [28] can write

$$
\left\| {}_aD_y^{\beta-2}(\mathscr{P}^+ P - P)\right\| \le ch^{k+1}.
$$

Combining this with Youngs inequality and we obtain

$$
\begin{aligned}
&\mathscr{M}\left(\mathscr{P}^- C - C, \mathscr{P}^+ H - \mathrm{H}, \mathscr{P}^+ P - P, \mathscr{P} Q - Q; \mathscr{P}^- e_C, \mathscr{P}^+ e_H, -\mathscr{P} e_Q, \mathscr{P}^+ e_P\right) \\
&\le \int_0^T \sum_{p=1}^{L} \left((\mathscr{P}^- C)_t - C_t, \mathscr{P}^- e_C\right)_{I_p} \mathrm{d}t + c_{T,a,b} h^{2k+2} + \frac{1}{c_{T,a,b}} \int_0^T \sum_{p=1}^{L} \|\mathscr{P} e_P\|_{I_p}^2 \,\mathrm{d}t \\
&\quad + \int_0^T \frac{\sqrt{b}\beta}{h} |(\mathscr{P}^- e_C)_{L+1}|^2 \mathrm{d}t + \int_0^T \sum_{p=1}^{L} \left\|\mathscr{P}^+ e_H\right\|_{I_p}^2 \mathrm{d}t,
\end{aligned}
$$

so the proof is complete. □

Theorem 4. *Assuming $C(y,t)$ be a exact solution of Equation* (1) *in $\Omega \subset \mathbb{R}$. assuming $C(y,t)$ be a exact solution to* (1) *which is sufficiently smooth in $\Omega \subset \mathbb{R}$. For small h, Suppose $C_h(y,t)$ be the numerical solution of the LDG method* (8)-(12)*, the error is estimated as follows:*

$$
\|C(y,t) - C_h(y,t)\| \le ch^{k+1},
$$

where c is a constant independent of h.

Proof. According to the lemma 1 and also $\|\mathscr{P}^- e_C(0)\| = 0$ we have

$$
\begin{aligned}
&\mathscr{M}\left(\mathscr{P}^- e_C, \mathscr{P}^+ e_H, \mathscr{P}^+ e_P, \mathscr{P} e_Q; \mathscr{P}^- e_C, \mathscr{P}^+ e_H, -\mathscr{P} e_Q, \mathscr{P}^+ e_P\right) \\
&\qquad = \frac{1}{2}\left\|\mathscr{P}^- e_C(T)\right\|^2 + \int_0^T \sum_{p=1}^{L} \left({}_aD_y^{\beta-2}\, \mathscr{P}^+ e_H, \mathscr{P}^+ e_H\right)_{I_p} \mathrm{d}t \\
&\qquad\quad + \int_0^T \sum_{p=1}^{L} \left\|\mathscr{P}^+ e_H\right\|_{I_p}^2 \mathrm{d}t + \int_0^T \frac{\sqrt{b}\beta}{h} |(\mathscr{P}^- e_C)_{L+1}|^2 \mathrm{d}t.
\end{aligned}
$$

Recalling Lemma 3, we have

$$
\begin{aligned}
&\frac{1}{2}\left\|\mathscr{P}^- e_C(T)\right\|^2 + \int_0^T \sum_{p=1}^{L} \left({}_aD_y^{\beta-2}\, \mathscr{P}^+ e_H, \mathscr{P}^+ e_H\right)_{I_p} \mathrm{d}t \\
&\le \int_0^T \sum_{p=0}^{L} \left((\mathscr{P}^- C)_t - C_t, \mathscr{P}^- e_C\right)_{I_p} \mathrm{d}t + c_{T,a,b} h^{2k+2} + \frac{1}{c_{T,a,b}} \int_0^T \sum_{p=0}^{L} \|\mathscr{P} e_H\|_{I_p}^2 \mathrm{d}t.
\end{aligned}
$$

Now using poincar Lemma in [28], we can write

$$
\frac{1}{2}\left\|\mathscr{P}^- e_C(T)\right\|^2 \le \int_0^T \sum_{p=0}^{L} \left((\mathscr{P}^- C)_t - C_t, \mathscr{P}^- e_C\right)_{I_p} \mathrm{d}t + c_{T,a,b} h^{2k+2}
$$

so the proof is complete. □

5. Numerical experiments

In this section, we demonstrate the validity of our analysis by giving three numerical examples. We show that in the LDG method, if the basic functions are B-Spline functions, the accuracy of the method is much higher. It should be noted that all numerical results have been done with the help of MATLAB software. We have measured the L^2 error

$$E_h = \left\| C(.,T) - \hat{C}_h(.,T) \right\|^2_{L^2(\mathbb{R})},$$

where $\hat{C}_h$ is the numerical solution which has been computed using a very fine grid h, the relative error

$$R_h = \left(\frac{1}{\left\| \hat{C}(.,T) \right\|^2_{L^2(\mathbb{R})}} \right) E_h,$$

and the approximate rate of convergence

$$\alpha_h = \left(\frac{1}{Log2} \right) \left(LogE_h - LogE_{\frac{h}{2}} \right).$$

Example 5.1. In the following nonlinear equation

$$\begin{aligned} &\tfrac{\partial C(y,t)}{\partial t} = \tfrac{\partial}{\partial y}\left(G\left(C\right) \tfrac{\partial C(y,t)}{\partial y} \right) + b \tfrac{\partial^\beta C(y,t)}{\partial y^\beta} + \rho(y,t), \quad (y,t) \in \mathbb{R} \times (0,T), \\ &C(y,0) = C_0(y), \quad y \in \mathbb{R}, \end{aligned}$$

we put

$$\begin{aligned} &G(C(y,t)) = C^2(y,t) - 2C(y,t) + 1, \\ &\rho(y,t) = e^{-3t} y^2 - y^{1-\beta} e^{-t} - y^{-\beta}, \\ &b = \Gamma(2-\beta) + \Gamma(1-\beta). \end{aligned}$$

The exact solution for $\beta \in (1,2)$ and $y \in (0,1)$ is $C(y,t) = e^{-t}(1-y)$. We have solved this problem for the basic functions of degrees 1, 2, and 3 when the β is very close to one and when the β is not too close to one. Tables 1 and 2 show the numerical results obtained.

Table 1: The LDG method for various k, L when β is not very close to 1 for Example 5.1 when $T = 1$.

		$k=1$			$k=2$			$k=3$		
β	L	E_h	α_h	R_h	E_h	α_h	R_h	E_h	α_h	R_h
1.3	16	8.22e-04	–	7.32e-05	3.35e-04	-	6.31e-06	4.23e-05	-	5.17e-07
	32	2.35e-04	1.80	6.43e-05	4.32e-05	2.95	7.33e-07	2.61e-06	4.05	5.63e-08
	64	5.12e-05	2.19	8.82e-07	5.25e-06	3.04	8.44e-08	1.55e-07	4.03	9.13e-09
1.5	16	8.35e-04	–	6.55e-06	2.38e-04	-	4.18e-06	2.03e-05	-	4.52e-07
	32	2.25e-04	1.89	7.43e-06	5.24e-05	3.04	8.03e-07	1.25e-06	4.02	6.61e-08
	64	5.14e-05	2.13	8.55e-07	6.44e-06	3.02	6.64e-08	7.50e-08	4.05	6.73e-09
1.8	16	8.72e-04	–	6.32e-05	1.26e-04	-	6.28e-06	3.11e-05	-	3.57e-07
	32	2.30e-04	1.92	7.45e-06	3.13e-05	2.95	8.03e-07	2.02e-06	3.95	2.63e-08
	64	5.48e-05	2.06	4.38e-05	3.83e-06	3.03	9.64e-08	1.22e-07	4.04	9.73e-09

Table 2: The LDG method for various k, L when β is very close to 1 for Example 5.1 when $T = 1$.

		$k=1$			$k=2$			$k=3$		
β	L	E_h	α_h	R_h	E_h	α_h	R_h	E_h	α_h	R_h
1.01	16	2.27e-04	-	7.44e-05	2.50e-06	-	5.22e-07	1.63e-06	-	5.17e-07
	32	5.71e-05	1.99	6.34e-06	3.15e-07	2.98	4.64e-08	1.02e-07	3.99	5.63e-08
	64	1.41e-05	2.01	5.33e-06	4.02e-08	2.97	5.24e-09	6.34e-09	4.00	9.13e-09
1.02	16	3.45e-04	-	8.34e-05	2.63e-06	-	6.48e-07	3.45e-07	-	7.32e-07
	32	8.56e-05	2.01	7.56e-06	3.18e-07	3.04	7.03e-08	2.12e-08	4.02	6.68e-08
	64	2.13e-05	2.00	5.33e-06	4.05e-08	2.97	8.54e-09	1.29e-09	4.03	7.73e-10
1.03	16	1.34e-04	-	9.21e-05	2.88e-06	-	5.33e-07	1.45e-07	-	5.81e-07
	32	3.36e-05	1.99	6.48e-06	3.62e-07	2.99	9.11e-08	9.00e-08	4.01	6.16e-08
	64	8.38e-06	2.00	5.54e-07	4.55e-08	2.99	9.04e-09	5.55e-10	4.02	8.23e-10

Example 5.2. We consider the following equation [32]

$$\frac{\partial C(y,t)}{\partial t} = \frac{\Gamma(6-\beta)}{\Gamma(6)} \frac{\partial^{\beta} C(y,t)}{\partial y^{\beta}} - e^{-t}(y^5 + y^{5-\beta}) \qquad y \in (0,1)\,, \beta \in [1,2].$$

$\Gamma(y)$ is the classic Gamma function. We consider the initial condition

$$C(y,0) = y^5,$$

and the Dirichlet boundary conditions

$$C(0,t) = 0, \quad C(1,t) = e^{-t},$$

The exact solution of the equation is $e^{-t}y^5$.

Table 3: Table c1(a) [32] when $k = 1$ for β is not very close to 1.

β	$L=2^5$	$L=2^6$		$L=2^7$		$L=2^8$	
	E_h	E_h	α_h	E_h	α_h	E_h	α_h
2.0	2.81e-04	7.82e-05	1.85	2.04e-05	1.94	5.20e-06	1.97
1.8	3.37e-04	9.37e-05	1.85	2.44e-05	1.94	6.14e-06	1.99
1.5	4.63e-04	1.27e-04	1.87	3.23e-05	1.98	7.93e-06	2.03
1.2	9.82e-04	3.17e-04	1.63	8.32e-05	1.93	2.02e-05	2.05

Example 5.3. Consider the following problem

$$\frac{\partial C(y,t)}{\partial t} + \frac{\partial}{\partial y}\left(\frac{C^2(y,t)}{2}\right) = \frac{\Gamma(4-\beta)}{\Gamma(4)} \frac{\partial^{\beta} C(y,t)}{\partial y} + e^{-t}y^3(-1 + 3e^{-t}y^2 - y^{-\beta}), \quad \text{in } [-0.1, 0.1] \times (0,1], \qquad (30)$$

with the discontinuous initial condition

$$C(y,0) = y^3,$$

Table 4: The LDG method by B-spline basic functions for β is not very close to 1 when $k = 1$.

β	$L = 2^5$	$L = 2^6$		$L = 2^7$		$L = 2^8$	
	E_h	E_h	α_h	E_h	α_h	E_h	α_h
2.0	3.21e-05	7.64e-06	2.07	1.97e-06	1.96	5.02e-07	1.97
1.8	4.37e-05	9.89e-06	2.04	2.42e-06	1.94	6.20e-07	1.96
1.5	3.61e-05	9.20e-06	1.97	2.30e-06	1.98	5.58e-07	2.04
1.2	2.45e-05	6.22e-06	1.97	1.51e-06	2.03	3.72e-07	2.02

and the Dirichlet boundary conditions

$$C(0,t) = 0, \quad C(1,t) = e^{-t},$$

Where $\beta \in [1, 2]$. The exact solution is $e^{-t}y^3$.

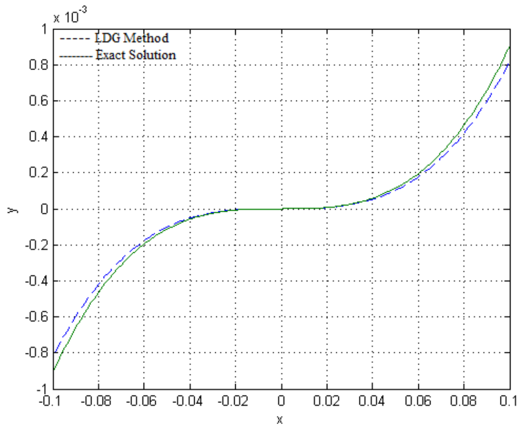
Conclusions

We have numerically solved the nonlinear fraction diffusion equation with fractional order derivative using the LDG method. In this article, we turn the problem into a higher-order device and then implement the Galerkin method. In this paper, we have used basic b-spline functions to solve this equation. In Galerkin's method, selecting the appropriate basic function can play an important role in achieving the exact answer. We have shown that the proposed LDG method is stable and has the order of h^{k+1} convergence. In the numerical results section, we conclude that the higher the degree of the basic function, the higher the accuracy of the method. In Section 5 in Example 2, we compare our method with reference 31 and show that our method is more accurate. The nonlinear equation has many applications in various sciences, especially physics in the fields of fluid mechanics, electromagnetism, thermodynamics and statistical mechanics, nuclear physics and biology. The LDG method transforms an equation into a system of simpler equations and approximates the solution of the equation by selecting the appropriate basic functions. In this article, we have solved the problem by choice b-spline functions as the basic functions. In this article, we have shown that the accuracy of the LDG method is very good by choice these functions.

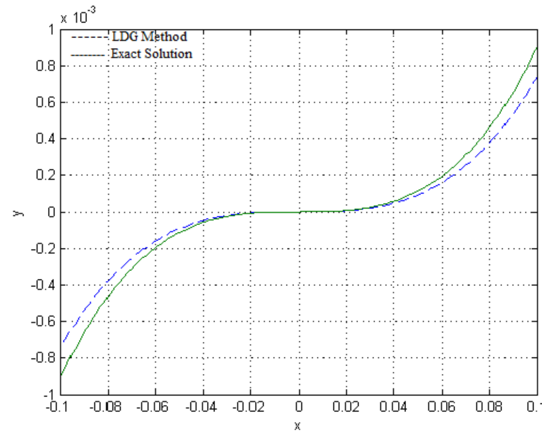
References

[1] A. C. Fowler, "Evolution equations for dunes and drumlins," *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales, Serie A Matematicas*, vol. 96, no. 3, pp. 377–387, 2002.

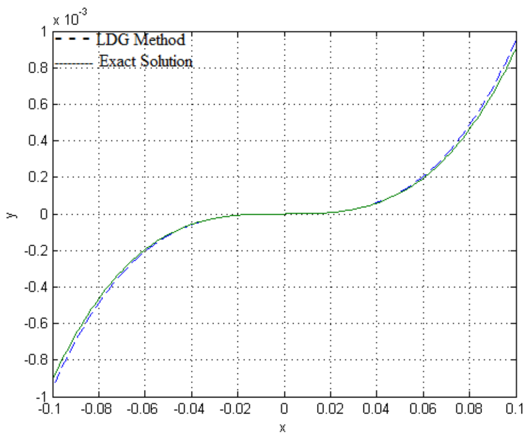
[2] H. Sun, Y. Zhang, D. Baleanu, W. Chen, and Y. Chen, "A new collection of real world applications of fractional calculus in science and engineering," *Communications in Nonlinear Science and Numerical Simulation*, vol. 64, pp. 213–231, 2018.



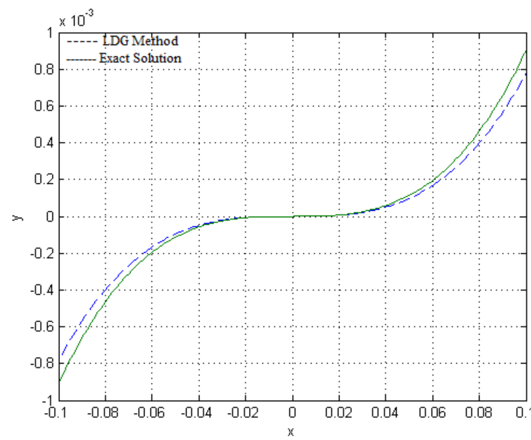
$(a)\quad \beta = 1.10$



$(a')\quad \beta = 1.70$



$(a)\quad \beta = 1.05$



$(a')\quad \beta = 1.40$

Figure 1: solutions of equation (30) with LDG method for h=1/160, T=1 and various β,

[3] R. Almeida, N. R. Bastos, and M. T. T. Monteiro, "Modeling some real phenomena by fractional differential equations," *Mathematical Methods in the Applied Sciences*, vol. 39, no. 16, pp. 4846–4855, 2016.

[4] M. Al-Smadi and O. A. Arqub, "Computational algorithm for solving fredholm time-fractional partial integrodifferential equations of dirichlet functions type with error estimates," *Applied Mathematics and Computation*, vol. 342, pp. 280 – 294, 2019.

[5] M. Saeedian, M. Khalighi, N. Azimi-Tafreshi, G. R. Jafari, and M. Ausloos, "Memory effects on epidemic evolution: The susceptible-infected-recovered epidemic model," *Physical Review E*, vol. 95, p. 022409, 2017.

[6] H. Safdari, M. Zare Kamali, A. Shirazi, M. Khalighi, G. Jafari, and M. Ausloos, "Fractional dynamics of network growth constrained by aging node interactions," *PLoS one*, vol. 11, pp. 1–13, 05 2016.

[7] M. M. Amirian, I. Towers, Z. Jovanoski, and A. J. Irwin, "Memory and mutualism in species sustainability: A time-fractional Lotka-Volterra model with harvesting," *Heliyon*, vol. 6, no. 9, p. e04816, 2020.

[8] S. Hosseinpour and A. Nazemi, "Solving fractional optimal control problems with fixed or free final states by Haar wavelet collocation method," *IMA Journal of Mathematical Control and Information*, vol. 33, no. 2, 2015.

[9] S. Ghasemi, A. Nazemi, and S. Hosseinpour, "Nonlinear fractional optimal control problems with neural network and dynamic optimization schemes," *Nonlinear Dynamics*, vol. 89, no. 4, 2017.

[10] M. Alabedalhadi, M. Al-Smadi, S. Al-Omari, D. Baleanu, and S. Momani, "Structure of optical soliton solution for nonliear resonant space-time Schrdinger equation in conformable sense with full nonlinearity term," *Physica Scripta*, vol. 95, p. 105215, sep 2020.

[11] M. Al-Smadi, O. A. Arqub, and S. Hadid, "Approximate solutions of nonlinear fractional Kundu-Eckhaus and coupled fractional massive Thirring equations emerging in quantum field theory using conformable residual power series method," *Physica Scripta*, vol. 95, p. 105205, sep 2020.

[12] M. A. Matlob and Y. Jamali, "The concepts and applications of fractional order differential calculus in modeling of viscoelastic systems: A primer," *Critical Reviews in Biomedical Engineering*, vol. 47, no. 4, 2019.

[13] E. C. de Oliveira and J. A. Tenreiro Machado, "A review of definitions for fractional derivatives and integral," *Math. Probl. Eng.*, vol. 2014, no. 5, p. 238459, 2014.

[14] M. Al-Smadi, O. A. Arqub, and S. Hadid, "An attractive analytical technique for coupled system of fractional partial differential equations in shallow water waves with conformable derivative," *Communications in Theoretical Physics*, vol. 72, p. 085001, jul 2020.

[15] R. Almeida, "A caputo fractional derivative of a function with respect to another function," *Communications in Nonlinear Science and Numerical Simulation*, vol. 44, pp. 460 – 481, 2017.

[16] J. Vanterler da C. Sousa and E. Capelas de Oliveira, "On the ψ-Hilfer fractional derivative," *Communications in Nonlinear Science and Numerical Simulation*, vol. 60, pp. 72 – 91, 2018.

[17] J. Vanterler da C. Sousa and E. Capelas de Oliveira, "Leibniz type rule: ψ-Hilfer fractional operator," *Communications in Nonlinear Science and Numerical Simulation*, vol. 77, pp. 305 – 311, 2019.

[18] J. V. d. C. Sousa, J. A. T. Machado, and E. C. de Oliveira, "The ψ-Hilfer fractional calculus of variable order and its applications," *Computational and Applied Mathematics*, vol. 39, p. 296, Oct 2020.

[19] D. S. Oliveira and E. C. de Oliveira, "Hilfer–Katugampola fractional derivatives," *Computational and Applied Mathematics*, vol. 37, pp. 3672–3690, Jul 2018.

[20] N. Alibaud, J. Droniou, and J. Vovelle, "Occurrence and non-appearance of shocks in fractal Burgers equations," *Journal of Hyperbolic Differential Equations*, vol. 4, no. 03, pp. 479–499, 2007.

[21] S. I. Muslih and O. P. Agrawal, "Riesz fractional derivatives and fractional dimensional space," *International Journal of Theoretical Physics*, vol. 49, no. 2, pp. 270–275, 2010.

[22] M. S. Espedal and K. H. Karlsen, "Numerical solution of reservoir flow models based on large time step operator splitting algorithms," in *Filtration in porous media and industrial application*, pp. 9–77, Springer, 2000.

[23] R. Cont and P. Tankov, *Financial modelling with jump processes.* Aug. 2004. 2003.

[24] M. Matalon, "Intrinsic flame instabilities in premixed and nonpremixed combustion," *Annu. Rev. Fluid Mech.*, vol. 39, pp. 163–191, 2007.

[25] N. Alibaud, "Entropy formulation for fractal conservation laws," *Journal of Evolution Equations*, vol. 7, no. 1, pp. 145–175, 2007.

[26] N. Alibaud, J. Droniou, and J. Vovelle, "Occurrence and non-appearance of shocks in fractal Burgers equations," *Journal of Hyperbolic Differential Equations*, vol. 4, no. 03, pp. 479–499, 2007.

[27] J. Droniou, "A numerical method for fractal conservation laws," *Mathematics of computation*, vol. 79, no. 269, pp. 95–124, 2010.

[28] Q. Xu and J. S. Hesthaven, "Discontinuous Galerkin method for fractional convection-diffusion equations," *SIAM Journal on Numerical Analysis*, vol. 52, no. 1, pp. 405–423, 2014.

[29] W. Liao, "A compact high-order finite difference method for unsteady convection-diffusion equation," *International Journal for Computational Methods in Engineering Science and Mechanics*, vol. 13, no. 3, pp. 135–145, 2012.

[30] M. Khalighi, M. Amirianmatlob, and A. Malek, "A new approach to solving multiorder time-fractional advectiondiffusionreaction equations using BEM and Chebyshev matrix," *Mathematical Methods in the Applied Sciences*, pp. 1–21, 2020.

[31] H. Egger and J. Schöberl, "A hybrid mixed discontinuous Galerkin finite-element method for convection–diffusion problems," *IMA Journal of Numerical Analysis*, vol. 30, no. 4, pp. 1206–1234, 2010.

[32] W. Deng and J. S. Hesthaven, "Local Discontinuous Galerkin methods for fractional diffusion equations," *ESAIM: Mathematical Modelling and Numerical Analysis*, vol. 47, no. 6, pp. 1845–1864, 2013.

[33] J. S. Hesthaven and T. Warburton, *Nodal discontinuous Galerkin methods: algorithms, analysis, and applications.* Springer Science & Business Media, 2007.

[34] S. Cifani, E. R. Jakobsen, and K. H. Karlsen, "The discontinuous Galerkin method for fractal conservation laws," *IMA journal of numerical analysis*, vol. 31, no. 3, pp. 1090–1122, 2011.

[35] K. Mustapha, B. Abdallah, K. Furati, and M. Nour, "A discontinuous Galerkin method for time fractional diffusion equations with variable coefficients," *Numerical Algorithms*, vol. 73, no. 2, pp. 517–534, 2016.

[36] A. Bouharguane and N. Seloula, "The local discontinuous Galerkin method for convection-diffusion-fractional anti-diffusion equations," *Applied Numerical Mathematics*, vol. 148, pp. 61 – 78, 2020.

[37] H. Richard, *Fractional calculus: an introduction for physicists.* World Scientific, 2014.

[38] B. Dumitru, D. Kai, and S. Enrico, *Fractional calculus: models and numerical methods*, vol. 3. World Scientific, 2012.