The Complexity of Power Graphs Associated With Finite Groups

3.6 Introduction

All graphs considered here are simple connected graphs. A *spanning tree* of a connected graph is just a subgraph that contains all the vertices and is a tree. Counting the number of spanning trees in a connected graph is a problem of long–standing interest in various fields of science. For a simple graph Γ, the number of spanning trees of Γ, denoted by $\kappa(\Gamma)$, is known as the *complexity* of Γ.

In this paper, we consider some graphs arising from finite groups. One well–known graph is the power graph, as defined more precisely below.

1.3.6 Definition. Let G be a finite group and X a nonempty subset of G. The *power graph* $\mathcal{P}(G, X)$, has X as its vertex set and two vertices x and y in X are joined by an edge if $\langle x\rangle \subseteq \langle y\rangle$ or $\langle y\rangle \subseteq \langle x\rangle$.

In the case $X = G$, we will simply write $\mathcal{P}(G)$ instead of $\mathcal{P}(G, G)$. Power graphs have been investigated by many authors in various contexts, see for instance [1, 12, 52]. Cleary, when $1 \in X$, the power graph is connected, and we can talk about the complexity of this graph. For convenience, we put $\kappa_G(X) = \kappa(\mathcal{P}(G, X))$ and $\kappa(G) = \kappa(\mathcal{P}(G))$.

A well known result due to Cayley [6] says that the complexity of the complete graph on n vertices is n^{n-2}. In [2] it was shown that a finite group has a complete power graph if and only if it is a cyclic p–group, where p is a prime number. Thus, as an immediate consequence