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Application of a Homogeneous Balance Method to Exact Solutions of Nonlinear Fractional Evolution Equations

The fractional Fan subequation method of the fractional Riccati equation is applied to construct the exact solutions of some nonlinear fractional evolution equations. In this paper, a powerful algorithm is developed for the exact solutions of the modified equal width equation, the Fisher equation, the nonlinear Telegraph equation, and the Cahn–Allen equation of fractional order. Fractional derivatives are described in the sense of the modified Riemann–Liouville derivative. Some relevant examples are investigated.
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1 Introduction

In recent years, fractional equations, both partial and ordinary ones, have been applied intensively in modeling many physical, engineering, chemistry, or biology complex phenomena [1–3]. Because the exact solutions of most of the nonlinear fractional partial differential equations (FPDEs) cannot be found easily, many attempts have been spent on achieving the exact solutions. As a result, many effective methods have been established such as the Bäcklund and Darboux transform [4], homotopy perturbation method [5], variational iteration method [6,7], homotopy analysis method [8,9], fractional subequation method [10,11], and so on [12–15].

The fractional subequation method is a very strong technique for finding exact solutions of nonlinear fractional differential equations. Recently, this method was introduced by Zhang and Zhang [16]. Further Guo et al. [17] and Lu [18] proposed the improved fractional subequation method to obtain the analytical solutions of nonlinear fractional differential equations. This method is determined on the homogeneous balance principle [19] and the modified Riemann–Liouville derivative described in Refs. [20,21]

$$D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-\xi)^{-\alpha}[f(\xi)-f(0)], \quad 0<\alpha\le 1 \tag{1}$$

and

$$D_x^\alpha f(x) = (f^{(\alpha-n)}(x))^{(n)}, \quad n\le\alpha<n+1, \quad n\ge 1 \tag{2}$$

We briefly present below some useful formulas of Jumarie's modified Riemann–Liouville derivative [20]:

$$D_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}x^{\gamma-\alpha}, \quad \gamma>0 \tag{3}$$

$$D_x^\alpha[f(x)g(x)] = g(x)D_x^\alpha f(x) + f(x)D_x^\alpha g(x) \tag{4}$$

$$D_x^\alpha f[g(x)] = f_g'[g(x)]D_x^\alpha g(x) = D_g^\alpha f[g(x)](g_x')^\alpha \tag{5}$$

The properties of the Jumarie derivative are suitable for changing the variables while the Riemann–Liouville gives complicated results. In this manuscript, we apply a fractional subequation for solving FPDEs that is based on the Bäcklund transformation technique [18] and known seed solutions.

The structure of this manuscript is as follows:

In Sec. 2, we give a brief exposition of the fractional subequation method. In Sec. 3, we present some applications of the proposed method to some nonlinear equations. A conclusion is shown in Sec. 4.

2 Summary of the Fractional Subequation Method

In this section, we present a brief summary of the fractional Fan subequation method for solving fractional differential equations [10]. Consider the following the fractional partial differential equation:

$$p\big(u, u_x, u_t, D_t^\alpha u, D_x^\alpha u, \ldots\big) = 0, \quad 0<\alpha\le 1 \tag{6}$$

where $D_t^\alpha u$ and $D_x^\alpha u$ are Jumarie's modified Riemann–Liouville derivatives of u, and x and t are two independent variables. Using the traveling wave transformation,

$$u(x,t) = u(\xi), \quad \xi = kx + ct \tag{7}$$

where c and k are constants, we rewrite Eq. (6) as

$$p\Big(u, ku', cu', k^\alpha D_\xi^\alpha u, c^\alpha D_\xi^\alpha u, \ldots\Big) = 0, \quad 0<\alpha\le 1 \tag{8}$$

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which is a nonlinear fractional ordinary differential equation. We assume that Eq. (8) has a solution as

$$u(\xi) = \sum_{i=0}^{n} a_i \psi^i \tag{9}$$

where a_i $(i = 0, 1, \ldots, n-1, n)$ are constants to be determined later, the positive integer n can be found by balancing the highest order derivatives with the nonlinear terms in Eqs. (6) or (8), and $\psi(\xi)$ is the Bäcklund transform for a fractional type Riccati equation

$$D_\xi^\alpha \varphi(\xi) = \sigma + \varphi^2(\xi), \quad 0 < \alpha \le 1 \tag{10}$$

in the following form:

$$\psi(\xi) = \frac{-\sigma B + D\varphi(\xi)}{D + B\varphi(\xi)} \tag{11}$$

where $B \neq 0$, D are arbitrary parameters, and $\psi(\xi)$ satisfies the following fractional Riccati equation:

$$D_\xi^\alpha \psi = \sigma + \psi^2, \quad 0 < \alpha \le 1 \tag{12}$$

Here $\varphi(\xi)$ denotes the known solutions of Eq. (10). Some exact solutions to Eq. (10) are derived by Zhang and Zhang [16] as follows:

$$\varphi(\xi) = \begin{cases} -\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}\xi), & \sigma < 0 \\ -\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\xi), & \sigma < 0 \\ \sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\xi), & \sigma > 0 \\ -\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\xi), & \sigma > 0 \\ -\dfrac{\Gamma(1+\alpha)}{\xi^\alpha + \omega}, \quad \omega = \text{const.}, & \sigma = 0 \end{cases} \tag{13}$$

By substituting Eq. (9) into Eq. (8) and equating each coefficient of $\psi(\xi)^k$ $(k = 0, 1, 2, \ldots)$ to zero, we obtain a set of algebraic equations for $c, k, a_i (i = 0, 1, \ldots, n-1, n)$. Using Mathematica software, we obtain $c, k, a_i (i = 0, 1, \ldots, n-1, n)$. Then the exact solution of Eq. (6) is obtained by replacing these values in Eq. (9). We mention that the limitation of this method is related to the range of validity of the properties of Jumarie's derivative, e.g., properties 3 and 4, respectively.

3 Applications

In this section, we apply the fractional subequation method for solving the modified fractional equal width equation, the fractional Fisher equation, the nonlinear fractional Telegraph equation, and the fractional Cahn–Allen equation.

3.1 Example. We consider the space-time fractional modified equal width equation as [22]

$$D_t^\alpha u = -u^2 D_x^\alpha + D_{xxt}^{3\alpha} \tag{14}$$

By using the traveling wave transformations

$$u = u(\xi), \quad \xi = kx + ct \tag{15}$$

Eq. (14) is reduced to a nonlinear fractional ordinal differential equation, namely,

$$c^\alpha D_\xi^\alpha u + k^\alpha u^2 D_\xi^\alpha u - k^{2\alpha} c^\alpha D_\xi^{3\alpha} u = 0 \tag{16}$$

By balancing $u^2 D_\xi^\alpha$ and $D_\xi^{3\alpha}$ gets $n = 1$, then we have the solution of Eq. (16) in the following form:

$$u = a_0 + a_1 \psi \tag{17}$$

Substituting Eq. (17) into Eq. (16) and setting coefficients of $\psi^j (j = 0, 1, \ldots, 4)$ to zero we finally obtain

$$a_0 = 0, \quad a_1 = \pm\sqrt{6c^\alpha k^\alpha}, \quad k^{2\alpha} = \frac{1}{2\sigma} \tag{18}$$

where σ denotes an arbitrary constant. Based on the above result and substituting Eq. (13) into Eq. (17), we have the exact solution of Eq. (14) as

$$u = \pm\sqrt{6c^\alpha k^\alpha}\frac{\sigma B + D\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}\xi)}{-D + B\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}\xi)}, \quad \sigma < 0$$

$$u = \pm\sqrt{6c^\alpha k^\alpha}\frac{\sigma B + D\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\xi)}{-D + B\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\xi)}, \quad \sigma < 0$$

$$u = \pm\sqrt{6c^\alpha k^\alpha}\frac{-\sigma B + D\sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\xi)}{D + B\sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\xi)}, \quad \sigma > 0$$

$$u = \pm\sqrt{6c^\alpha k^\alpha}\frac{\sigma B + D\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\xi)}{-D + B\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\xi)}, \quad \sigma > 0$$

3.2 Example. Let us consider the space-time fractional Fisher equation [22]

$$D_t^\alpha u = D_{xx}^{2\alpha} + 2u(1-u^2) + \mu(1-u^2) \tag{19}$$

As before, by using traveling wave transformation we end up with a nonlinear fractional differential equation, namely,

$$c^\alpha D_\xi^\alpha u - k^{2\alpha} D_\xi^{2\alpha} u - 2u(1-u^2) - \mu(1-u^2) = 0 \tag{20}$$

Like in the previous example, we obtain the following set of solutions.

If $\sigma \neq 0$

Case 1. Consider

$$a_0 = \frac{2(\sigma k^{2\alpha} + 1)(48\sigma k^{2\alpha} + 5\mu^2 + 12)}{\mu(\mu^2 - 36)}, \quad a_1 = \pm k^\alpha$$
$$k^\alpha = \pm\frac{\sqrt{-\dfrac{6\mu^2(\sigma k^{2\alpha} + 4) \pm 36\mu\sqrt{12\sigma k^{2\alpha} + \mu^2 + 12} \mp \mu^3\sqrt{12\sigma k^{2\alpha} + \mu^2 + 12} + 72(-\sigma^2 k^{4\alpha} + \sigma k^{2\alpha} + 2) + \mu^4}{\sigma(12\sigma k^{2\alpha} + \mu^2 + 12)}}}{3\sqrt{6}} \tag{21}$$

Case 2. Consider

$$a_0=\frac{1}{6}k^{-\alpha}(-c^\alpha-\mu k^\alpha),\quad a_1=k^\alpha,\quad \mu=\pm 6$$
$$k^\alpha=\pm\frac{\sqrt{\sigma k^{2\alpha}-8}}{3\sqrt{\sigma}},\quad c^\alpha=\pm\frac{2\sqrt{\sigma k^{4\alpha}+28k^{2\alpha}}}{\sqrt{3}} \tag{22}$$

Case 3. Consider

$$a_0=\frac{1}{6}k^{-\alpha}(c^\alpha-\mu k^\alpha),\quad a_1=-k^\alpha,\quad \mu=\pm 6$$
$$k^\alpha=\pm\frac{\sqrt{\sigma k^{2\alpha}-8}}{3\sqrt{\sigma}},\quad c^\alpha=\pm\frac{2\sqrt{\sigma k^{4\alpha}+28k^{2\alpha}}}{\sqrt{3}} \tag{23}$$

if $\sigma=0$.

Case 4. We have

$$a_0=\frac{1}{32}\mu\left(\mu^2-20\right),\quad a_1=\pm k^\alpha,\quad \mu=\pm 2,$$
$$c^\alpha=\pm\frac{1}{16}\left(44\mu k^\alpha-3\mu^3k^\alpha\right) \tag{24}$$

Case 1 leads us to the following solutions of Eq. (19):

$$u=\frac{2(\sigma k^{2\alpha}+1)(48\sigma k^{2\alpha}+5\mu^2+12)}{\mu(\mu^2-36)}\pm k^\alpha\frac{\sigma B+D\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}\xi)}{-D+B\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}\xi)},\quad \sigma<0$$
$$u=\frac{2(\sigma k^{2\alpha}+1)(48\sigma k^{2\alpha}+5\mu^2+12)}{\mu(\mu^2-36)}\pm k^\alpha\frac{\sigma B+D\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\xi)}{-D+B\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\xi)},\quad \sigma<0$$
$$u=\frac{2(\sigma k^{2\alpha}+1)(48\sigma k^{2\alpha}+5\mu^2+12)}{\mu(\mu^2-36)}\pm k^\alpha\frac{-\sigma B+D\sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\xi)}{D+B\sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\xi)},\quad \sigma>0$$
$$u=\frac{2(\sigma k^{2\alpha}+1)(48\sigma k^{2\alpha}+5\mu^2+12)}{\mu(\mu^2-36)}\pm k^\alpha\frac{\sigma B+D\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\xi)}{-D+B\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\xi)},\quad \sigma>0$$

From the second, we can obtain many other exact solutions of Eq. (19), such as

$$u=\frac{1}{6}k^{-\alpha}(-c^\alpha+6k^\alpha)\pm k^\alpha\frac{\sigma B+D\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}\xi)}{-D+B\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}\xi)},\quad \sigma<0$$
$$u=\frac{1}{6}k^{-\alpha}(-c^\alpha+6k^\alpha)\pm k^\alpha\frac{\sigma B+D\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\xi)}{-D+B\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\xi)},\quad \sigma<0$$
$$u=\frac{1}{6}k^{-\alpha}(-c^\alpha+6k^\alpha)\pm k^\alpha\frac{-\sigma B+D\sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\xi)}{D+B\sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\xi)},\quad \sigma>0$$
$$u=\frac{1}{6}k^{-\alpha}(-c^\alpha+6k^\alpha)\pm k^\alpha\frac{\sigma B+D\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\xi)}{-D+B\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\xi)},\quad \sigma>0$$

From the third, we can obtain many other exact solutions of Eq. (19), such as

$$u=\frac{1}{6}k^{-\alpha}(+c^\alpha-6k^\alpha)\pm k^\alpha\frac{\sigma B+D\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}\xi)}{-D+B\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}\xi)},\quad \sigma<0$$
$$u=\frac{1}{6}k^{-\alpha}(+c^\alpha-6k^\alpha)\pm k^\alpha\frac{\sigma B+D\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\xi)}{-D+B\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\xi)},\quad \sigma<0$$
$$u=\frac{1}{6}k^{-\alpha}(+c^\alpha-6k^\alpha)\pm k^\alpha\frac{-\sigma B+D\sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\xi)}{D+B\sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\xi)},\quad \sigma>0$$
$$u=\frac{1}{6}k^{-\alpha}(+c^\alpha-6k^\alpha)\pm k^\alpha\frac{\sigma B+D\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\xi)}{-D+B\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\xi)},\quad \sigma>0$$

Finally, the fourth case provides the following exact solutions of Eq. (19)

$$u=\ \pm 1\pm k^\alpha\frac{\sigma B(\xi^\alpha+\omega)+D\Gamma(1+\alpha)}{-D(\xi^\alpha+\omega)+B\Gamma(1+\alpha)},\quad \sigma=0$$

3.3 Example. We consider the nonlinear space-time fractional Telegraph equation [22]

$$D_{tt}^{2\alpha}u-D_{xx}^{2\alpha}+D_t^\alpha+\gamma u+\beta u^3=0 \tag{25}$$

Using the transformations of the Sec. 3.2, the transformed equation of Eq. (25) becomes

$$c^{2\alpha}D_\xi^{2\alpha}u-k^{2\alpha}D_\xi^{2\alpha}+c^\alpha D_\xi^\alpha+\gamma u+\beta u^3=0 \tag{26}$$

Similarly, we obtain

$$a_0=\ \pm\ \frac{i\sqrt{\gamma}}{2\sqrt{\beta}},\quad a_1=\ \pm\ \frac{2ic^\alpha}{3\sqrt{\beta}\sqrt{\gamma}},$$
$$c^\alpha=\ \pm\ \frac{3\sqrt{\gamma c^{2\alpha}-\gamma k^{2\alpha}}}{\sqrt{2}},\quad \sigma=-\frac{\gamma}{c^{2\alpha}-k^{2\alpha}} \tag{27}$$

By using the above results, we have the exact solution in the form

$$u=\pm\frac{i\sqrt{\gamma}}{2\sqrt{\beta}}\pm\frac{2ic^\alpha}{3\sqrt{\beta}\sqrt{\gamma}}\frac{\sigma B+D\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}\xi)}{-D+B\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}\xi)},\quad \sigma<0$$
$$u=\pm\frac{i\sqrt{\gamma}}{2\sqrt{\beta}}\pm\frac{2ic^\alpha}{3\sqrt{\beta}\sqrt{\gamma}}\frac{\sigma B+D\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\xi)}{-D+B\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\xi)},\quad \sigma<0$$
$$u=\pm\frac{i\sqrt{\gamma}}{2\sqrt{\beta}}\pm\frac{2ic^\alpha}{3\sqrt{\beta}\sqrt{\gamma}}\frac{-\sigma B+D\sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\xi)}{D+B\sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\xi)},\quad \sigma>0$$
$$u=\pm\frac{i\sqrt{\gamma}}{2\sqrt{\beta}}\pm\frac{2ic^\alpha}{3\sqrt{\beta}\sqrt{\gamma}}\frac{\sigma B+D\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\xi)}{-D+B\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\xi)},\quad \sigma>0$$

3.4 Example. Consider the space-time fractional Cahn–Allen equation [22]

$$D_t^\alpha u=D_{xx}^{2\alpha}-u^3+u \tag{28}$$

Using the transformations explained in the previous section, the transformed equation of Eq.(28) becomes

$$c^\alpha D_\xi^\alpha u-k^{2\alpha}D_\xi^{2\alpha}+u^3-u=0 \tag{29}$$

Thus, we have the following results:

$$a_0 = \pm \frac{1}{2}, \quad a_1 = \mp \frac{2c^\alpha}{3}, \quad c^\alpha = \pm \frac{3\sqrt{k^{2\alpha}}}{\sqrt{2}}, \quad k^{2\alpha} = -\frac{1}{8\sigma} \tag{30}$$

Therefore, we obtain the exact solution as

$$u = \pm \tfrac{1}{2} \mp \frac{2c^\alpha}{3} \frac{\sigma B + D\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}\xi)}{-D + B\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}\xi)}, \quad \sigma < 0$$

$$u = \pm \frac{1}{2} \mp \frac{2c^\alpha}{3} \frac{\sigma B + D\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\xi)}{-D + B\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\xi)}, \quad \sigma < 0$$

$$u = \pm \frac{1}{2} \mp \frac{2c^\alpha}{3} \frac{-\sigma B + D\sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\xi)}{D + B\sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\xi)}, \quad \sigma > 0$$

$$u = \pm \frac{1}{2} \mp \frac{2c^\alpha}{3} \frac{\sigma B + D\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\xi)}{-D + B\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\xi)}, \quad \sigma > 0$$

4 Conclusions

In this manuscript, based on the Bäcklund transformation of the fractional Riccati equation with known solutions, we have obtained the exact solutions of four nonlinear fractional differential equations having many practical applications. Mathematica has been used for computations and programming in this manuscript. The results show that this method is accurate and effective, and it can be used for many other nonlinear fractional differential equations with real world applications.

References

[1] Miller, K. S., and Ross, B., 1993, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York.
[2] Kilbas, A. A., Srivastava, H. M., and Trujillo, J. J., 2006, *Theory and Applications of Fractional Differential Equations*, Elsevier, New York.
[3] Podlubny, I., 1999, *Fractional Differential Equations*, Academic Press, New York.
[4] Miurs, M. R., 1978, *Bäcklund Transformation*, Springer, Berlin.
[5] Jafari, H., Kadkhoda, N., Tajadodi, H., and Hosseini Matikolai, S. A., 2010, "Homotopy Perturbation Pade Technique for Solving Fractional Riccati Differential Equations," Int. J. Nonlinear Sci. Numer. Simul., **11**, pp. 271–275.
[6] Jafari, H., and Tajadodi, H., 2010, "He's Variational Iteration Method for Solving Fractional Riccati Differential Equation," Int. J. Differ. Equ., **2010**, p. 764738.
[7] Jafari, H., Tajadodi, H., and Baleanu, D., 2013, "A Modified Variational Iteration Method for Solving Fractional Riccati Differential Equation by Adomian Polynomials," Fractional Calculus Appl. Anal., **16**(1), pp. 109–122.
[8] Jafari, H., Das, S., and Tajadodi, H., 2011, "Solving a Multi-Order Fractional Differential Equation Using Homotopy Analysis Method," J. King Saud Univ., Sci., **23**(2), pp. 151–155.
[9] Jafari, H., Tajadodi, H., and Biswas, A., 2011, "Homotopy Analysis Method for Solving a Couple of Evolution Equations and Comparison With Adomian's Decomposition Method," Waves Random Complex Media, **21**(4), pp. 657–667.
[10] Jafari, H., Tajadodi, H., Kadkhoda, N., and Baleanu, D., 2013, "Fractional Sub-Equation Method For Cahn-Hilliard And Klein-Gordon Equations," Abstr. Appl. Anal., **2013**, p. 587179.
[11] Zhao, J., Tang, B., Kumar, S., and Hou, Y., 2012, "The Extended Fractional Subequation Method for Nonlinear Fractional Differential Equations," Math. Probl. Eng., **2012**, p. 924956.
[12] Agrawal, O. P., Hasan, M. M., and Tangpong, X. W., 2012 "A Numerical Scheme for a Class of Parametric Problem of Fractional Variational Calculus," ASME J. Comput. Nonlinear Dyn., **7**(1), p. 021005.
[13] Evirgen, F., and Özdemir, N., 2010, "Multistage Adomian Decomposition Method for Solving NLP Problems Over a Nonlinear Fractional Dynamical System," ASME J. Comput. Nonlinear Dyn., **6**(2), p. 021003.
[14] Gafiychuk, V., and Datsko, B., 2012, "Different Types of Instabilities and Complex Dynamics in Reaction-Diffusion Systems With Fractional Derivatives," ASME J. Comput. Nonlinear Dyn. **7**(3), p. 031001.
[15] Kumar, P., and Agrawal, O. P., 2005, "Numerical Scheme for the Solution of Fractional Differential Equations of Order Greater Than One," ASME J. Comput. Nonlinear Dyn., **1**(2), pp. 178–185.
[16] Zhang, S., and Zhang, H. Q., 2011, "Fractional Sub-Equation Method and its Applications to Nonlinear Fractional PDEs," Phys. Lett. A, **375**(7), pp. 1069–1073.
[17] Guo, S., Mei, L., Li, Y., and Sun, Y., 2012, "The Improved Fractional Sub-Equation Method and its Applications to the Space-Time Fractional Differential Equations in Fluid Mechanics," Phys. Lett. A, **376**(4), pp. 407–411.
[18] Lu, B., 2012. "Bäcklund Transformation of Fractional Riccati Equation and its Applications to Nonlinear Fractional Partial Differential Equations," Phys. Lett. A, **376**(28–29) pp. 2045–2048.
[19] Wang, M., 1995, "Solitary Wave Solutions for Variant Boussinesq Equations," Phys. Lett. A, **199**(3–4), pp. 169–172.
[20] Jumarie, G., 2006, "Modified Riemann–Liouville Derivative and Fractional Taylor Series of Nondifferentiable Functions Further Results," Comput. Math. Appl., **51**, pp. 1367–1376.
[21] Jumarie, G., 2010, "Cauchy's Integral Formula via the Modified Riemann-Liouville Derivative for Analytic Functions of Fractional Order," Appl. Math. Lett., **23**, pp. 1444–1450.
[22] Taghizadeh, N., Mirzazadeh, M., Samiei Paghaleh, A., Vahidi, J., 2012, "Exact Solutions of Nonlinear Evolution Equations by Using the Modified Simple Equation Method," Ain Shams Eng. J., **3**(12), pp. 321–325.