The Cubic Equation Formula

Theorem (The Cubic Equation Formula). *The roots of the cubic equation*

$$ax^3 + bx^2 + cx + d = 0 \tag{1}$$

where $a \neq 0$, *can be computed by*

$$x = \sqrt[3]{-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a} \pm \sqrt{\left(\frac{b^3}{27a^3} - \frac{bc}{6a^2} + \frac{d}{2a}\right)^2 + \left(-\frac{b^2}{9a^2} + \frac{c}{3a}\right)^3}}$$
$$+ \frac{b^2 - 3ac}{9a^2 \sqrt[3]{-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a} \pm \sqrt{\left(\frac{b^3}{27a^3} - \frac{bc}{6a^2} + \frac{d}{2a}\right)^2 + \left(-\frac{b^2}{9a^2} + \frac{c}{3a}\right)^3}}} - \frac{b}{3a}.$$

Proof. If we divide everything on the left hand side and the right hand side of $=$ in Equation (1) by a, we get the cubic equation

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$$

So the cubic Equation (1) can be reduced to the cubic equation

$$x^3 + Bx^2 + Cx + D = 0 \tag{2}$$

where

$$B = \frac{b}{a}, \qquad C = \frac{c}{a}, \qquad D = \frac{d}{a}. \tag{3}$$

We can even reduce the cubic Equation (2) by eliminating the x^2 term. Let $x = y + \alpha$. Then

$$\begin{aligned} x^3 &= (y+\alpha)^3 = y^3 + 3\alpha y^2 + 3\alpha^2 y + \alpha^3, \\ Bx^2 &= B(y+\alpha)^2 = B(y^2 + 2\alpha y + \alpha^2) = By^2 + 2\alpha By + \alpha^2 B, \\ Cx &= C(y+\alpha) = Cy + \alpha C. \end{aligned}$$

So the cubic Equation (2) can be written as

$$y^3 + (3\alpha + B)y^2 + (3\alpha^2 + 2\alpha B + C)y + (\alpha^3 + \alpha^2 B + \alpha C + D) = 0$$

We wish to eliminate the y^2 term so we want $3\alpha + B = 0$. This gives $\alpha = -\frac{1}{3}B$. It can be computed that

$$\begin{aligned} 3\alpha^2 + 2\alpha B + C &= 3\left(-\frac{1}{3}B\right)^2 + 2\left(-\frac{1}{3}B\right)B + C \\ &= 3 \cdot \frac{1}{9}B^2 - \frac{2}{3}B^2 + C \\ &= \frac{1}{3}B^2 - \frac{2}{3}B^2 + C \\ &= -\frac{1}{3}B^2 + C \end{aligned}$$

and

$$\begin{aligned} \alpha^3 + \alpha^2 B + \alpha C + D &= \left(-\frac{1}{3}B\right)^3 + \left(-\frac{1}{3}B\right)^2 B - \frac{1}{3}BC + D \\ &= -\frac{1}{27}B^3 + \frac{1}{9}B^3 - \frac{1}{3}BC + D \\ &= \frac{2}{27}B^3 - \frac{1}{3}BC + D. \end{aligned}$$

Hence, the cubic Equation (2) can be reduced by the substitution

$$x = y - \frac{1}{3}B \tag{4}$$

to the cubic equation

$$y^3 + py + q = 0 \tag{5}$$

where

$$p = -\frac{1}{3}B^2 + C, \qquad q = \frac{2}{27}B^3 - \frac{1}{3}BC + D. \tag{6}$$

So the problem of finding the roots of the cubic Equation (1) has reduced to finding the roots of the cubic Equation (5) which is easier to deal with.

So we are now trying to solve the cubic Equation (5). Let $y = z - \frac{p}{3z}$. Then

$$\begin{aligned} y^3 &= \left(z - \frac{p}{3z}\right)^3 = z^3 - 3 \cdot \frac{p}{3z} z^2 + 3 \cdot \frac{p^2}{9z^2} \cdot z - \frac{p^3}{27z^3} = z^3 - pz + \frac{p^2}{3z} - \frac{p^3}{27z^3}, \\ py &= p\left(z - \frac{p}{3z}\right) = pz - \frac{p^2}{3z}. \end{aligned}$$

So the cubic Equation (5) by the substitution

$$y = z - \frac{p}{3z} \tag{7}$$

reduces to

$$z^3 - \frac{p^3}{27z^3} + q = 0 \tag{8}$$

Multiplying Equation (8) by $27z^3$, gives the quadratic equation

$$27(z^3)^2 + 27qz^3 - p^3 = 0$$

in z^3. We can compute z^3 by using the quadratic equation formula.

$$\begin{aligned} z^3 &= \frac{-27q \pm \sqrt{(27q)^2 - 4(27)(-p^3)}}{2 \cdot 27} \\ &= \frac{-27q \pm \sqrt{(27q)^2 + 4 \cdot 27p^3}}{2 \cdot 27} \\ &= -\frac{27q}{2 \cdot 27} \pm \sqrt{\frac{27^2 q^2}{2^2 \cdot 27^2} + \frac{4 \cdot 27p^3}{2^2 \cdot 27^2}} \\ &= -\frac{1}{2}q \pm \sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3} \end{aligned}$$

Hence, we can compute z.

$$z = \sqrt[3]{-\frac{1}{2}q \pm \sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3}} \tag{9}$$

By using Equation (6), we have

$$\frac{1}{2}q = \frac{1}{2}\left(\frac{2}{27}B^3 - \frac{1}{3}BC + D\right) = \frac{1}{27}B^3 - \frac{1}{6}BC + \frac{1}{2}D,$$

and

$$\frac{1}{3}p = \frac{1}{3}\left(-\frac{1}{3}B^2 + C\right) = -\frac{1}{9}B^2 + \frac{1}{3}C$$

By substituting Equation (9) into Equation (7), we get

$$y = \sqrt[3]{-\frac{1}{27}B^3 + \frac{1}{6}BC - \frac{1}{2}D \pm \sqrt{\left(\frac{1}{27}B^3 - \frac{1}{6}BC + \frac{1}{2}D\right)^2 + \left(-\frac{1}{9}B^2 + \frac{1}{3}C\right)^3}} + \frac{\frac{1}{3}B^2 - C}{3\sqrt[3]{-\frac{1}{27}B^3 + \frac{1}{6}BC - \frac{1}{2}D \pm \sqrt{\left(\frac{1}{27}B^3 - \frac{1}{6}BC + \frac{1}{2}D\right)^2 + \left(-\frac{1}{9}B^2 + \frac{1}{3}C\right)^3}}}$$

We can write Equation (4) as

$$x = \sqrt[3]{-\frac{1}{27}B^3 + \frac{1}{6}BC - \frac{1}{2}D \pm \sqrt{\left(\frac{1}{27}B^3 - \frac{1}{6}BC + \frac{1}{2}D\right)^2 + \left(-\frac{1}{9}B^2 + \frac{1}{3}C\right)^3}} + \frac{\frac{1}{3}B^2 - C}{3\sqrt[3]{-\frac{1}{27}B^3 + \frac{1}{6}BC - \frac{1}{2}D \pm \sqrt{\left(\frac{1}{27}B^3 - \frac{1}{6}BC + \frac{1}{2}D\right)^2 + \left(-\frac{1}{9}B^2 + \frac{1}{3}C\right)^3}}} - \frac{1}{3}B$$

We can finally give an explicit formula for the roots of the cubic Equation (1) by using Equation (3).

$$x = \sqrt[3]{-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a} \pm \sqrt{\left(\frac{b^3}{27a^3} - \frac{bc}{6a^2} + \frac{d}{2a}\right)^2 + \left(-\frac{b^2}{9a^2} + \frac{c}{3a}\right)^3}} + \frac{b^2 - 3ac}{9a^2\sqrt[3]{-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a} \pm \sqrt{\left(\frac{b^3}{27a^3} - \frac{bc}{6a^2} + \frac{d}{2a}\right)^2 + \left(-\frac{b^2}{9a^2} + \frac{c}{3a}\right)^3}}} - \frac{b}{3a}$$

This completes the proof. □

References

[1] J. Bienaymé, "Considérations à l'appui de la découverte de Laplace sur la loi de probabilité dans la méthode des moindres carrés," *Comptes Rendus hebdomadaires des séances de l'Académie des Sciences* (Paris) **37** (1853), 309–324.

[2] Herman Chernoff, "A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations," *Annals of Mathematical Statistics* **23** (1952), 493–507.

[3] D. Gale and L. S. Shapley, "College admissions and the stability of marriage," *American Mathematical Monthly* **69** (1962), 9–15.