Lemma 1. Let $x \in G$ such that $|E^2_{\frac{G}{\mathbb{R}_2}}(x\mathbb{R}_2)| = P$, P a prime where $\mathbb{R}_2 = \mathbb{R}_2(G)$.Then for all $y \in G$ with $E^2_{\frac{G}{\mathbb{R}_2}}(x\mathbb{R}_2) = E^2_{\frac{G}{\mathbb{R}_2}}(y\mathbb{R}_2)$, we have $E^2_G(x) = E^2_G(y)$

Proof. We have that $\frac{E^2_G(x)}{\mathbb{R}_2} \leq E^2_{\frac{G}{\mathbb{R}_2}}(x\mathbb{R}_2)$. Suppose that $\frac{E^2_G(x)}{\mathbb{R}_2} < E^2_{\frac{G}{\mathbb{R}_2}}(x\mathbb{R}_2)$ since $|E^2_{\frac{G}{\mathbb{R}_2}}(x\mathbb{R}_2)| = P$ and $|\frac{E^2_G(x)}{\mathbb{R}_2}|$ divides $|E^2_{\frac{G}{\mathbb{R}_2}}(x\mathbb{R}_2)|$ so $|\frac{E^2_G(x)}{\mathbb{R}_2}| = 1 \Longrightarrow E^2_G(x) = \mathbb{R}_2 \Longrightarrow x \in \mathbb{R}_2$, a contradiction clearly , $\frac{E^2_G(y)}{\mathbb{R}_2} \leq E^2_{\frac{G}{\mathbb{R}_2}}(y\mathbb{R}_2) = E^2_{\frac{G}{\mathbb{R}_2}}(x\mathbb{R}_2)$. Therefore $|E^2_{\frac{G}{\mathbb{R}_2}}(x\mathbb{R}_2)| = |\frac{E^2_G(y)}{\mathbb{R}_2}|$ and so $\frac{E^2_G(y)}{\mathbb{R}_2} = \frac{E^2_G(x)}{\mathbb{R}_2}$. Thus

$$\frac{E^2_G(x)}{\mathbb{R}_2} = \frac{E^2_G(y)}{\mathbb{R}_2} = \{\mathbb{R}_2, t_1\mathbb{R}_2, t_2\mathbb{R}_2, \ldots, t_{p-1}\mathbb{R}_2\}$$

where$\{t_1, \ldots, t_{p-1}\} \in E^2_G(x) \bigcap E^2_G(y) - \mathbb{R}_2$. So $E^2_G(x) = E^2_G(y)$. Hence , the lemma follows. □

Theorem 1. Let G be a group and P be a prime. If $\frac{G}{\mathbb{R}_2(G)} \cong C_P \times C_P$ then $|E^2_G(x)| = P + 2$

Proof. Suppose, first that $\frac{G}{\mathbb{R}_2(G)} \cong C_P \times C_P$ then

$$\begin{aligned} \frac{G}{\mathbb{R}_2(G)} &= \left\langle \mathbb{R}_2x, \mathbb{R}_2y | (\mathbb{R}_2x)^P = (\mathbb{R}_2y)^P = \mathbb{R}_2 \right\rangle, (\mathbb{R}_2x)(\mathbb{R}_2y) = (\mathbb{R}_2y)(\mathbb{R}_2x) \\ &= \left\langle \mathbb{R}_2x, \mathbb{R}_2y | x^P, y^P, x^{-1}y^{-1}xy \in \mathbb{R}_2 \right\rangle \end{aligned}$$

If $\frac{H}{\mathbb{R}_2} < \frac{G}{\mathbb{R}_2}$, then $\left| \frac{\frac{G}{\mathbb{R}_2}}{\frac{H}{\mathbb{R}_2}} \right| = P \Longrightarrow \left| \frac{H}{\mathbb{R}_2} \right| = P$. There for $H = \mathbb{R}_2 \bigcup \mathbb{R}_2 t_2 \bigcup \cdots \bigcup \mathbb{R}_2 t_{p-1}$ where $t_i \in H - \mathbb{R}_2$ and $i \in \{1, 2, \ldots, p-1\}$ so the proper subgroups of G properly containing $\mathbb{R}_2$ are

$$\begin{aligned} H_1 =& \mathbb{R}_2 \bigcup \mathbb{R}_2x^2 \bigcup \mathbb{R}_2x^3 \bigcup \cdots \bigcup \mathbb{R}_2x^{P-1} \\ H_2 =& \mathbb{R}_2 \bigcup \mathbb{R}_2y \bigcup \mathbb{R}_2y^2 \bigcup \ldots \mathbb{R}_2y^{P-1} \\ H_3 =& \mathbb{R}_2 \bigcup \mathbb{R}_2x^2y \bigcup \mathbb{R}_2x^2y^2 \bigcup \mathbb{R}_2x^2y^{P-1} \\ &\vdots \\ H_{P+1} =& \mathbb{R}_2 \bigcup \mathbb{R}_2x^{P-1}y \bigcup \mathbb{R}_2x^{P-1}y^2 \bigcup \ldots \mathbb{R}_2x^{P-1}y^{P-1} \end{aligned}$$

No we will show that $H_1, H_2, \ldots H_{P+1}$ are only proper Engelizer of G. Let $a \in G - \mathbb{R}_2$, $a \in G - \mathbb{R}_2$ then $\mathbb{R}_2a = \mathbb{R}_2k$ such that

$$k \in \left\{x, \ldots, x^{P-1}, y, \ldots, y^{P-1}, xy, xy^2, \ldots, xy^{P-1}, \ldots x^{P-1}y, \ldots, x^{P-1}y^{P-1}\right\}.$$

There for $E^2_{\frac{G}{\mathbb{R}_2}}(\mathbb{R}_2 a) = E^2_{\frac{G}{\mathbb{R}_2}}(\mathbb{R}_2 k)$ from lemma 1 we have $E^2_G(a) = E^2_G(k)$. Again let $k \in H_i - \mathbb{R}_2(G)$ then $E^2_G(k) \in \bigcup_{j=1}^{P+1} H_j$ as $H - 1, H_2, \ldots H_{P+1}$ are the only proper subgroups of G. Also $k \in E^2_G(k)$, therefor $E^2_G(k) \neq H_j, 1 \leq j \leq P + 1$ and $i \neq j$. therefore $E^2_G(k) = H_i$. Hence $H_1, H_2, \ldots H_{P+1}$ are the only proper Engelizer of G. thus $|E^2_G| = P + 2$. This complete the proof. To the reverse of theorem we check for $p = 2, 3, 5, 7$. □

Theorem 2. Let G be agroup. Then $|Cent(G)| = 4$ if only if $\frac{G}{R_2(G)} \cong C_2 \times C_2$

Proof. 1 □