ON THE EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF INFINITE SEMIPOSITONE PROBLEMS

We discuss the existence of a positive solution to the infinite semipositone problem

$$-\Delta u = -au + bu^2 - du^3 - f(u) - \frac{c}{u^\alpha}\,, \quad x \in \Omega\,, \quad u = 0\,, \quad x \in \partial\Omega$$

where $\alpha \in (0, 1)$, a, b, d and c are positive constants, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, Δ is the Laplacian operator, and $f : [0, \infty) \to \mathbb{R}$ is a nondecreasing continuous function such that $f(u) \to \infty$ and $f(u)/u \to 0$ as $u \to \infty$. We obtain our result via the method of sub- and supersolutions. We also extend our result to classes of infinite semipositone system and p-Laplacian problem.

Keywords: Positive solution; Infinite semipositone; Sub- and supersolutions
MSC2010: 35J61, 35J66

1 Introduction

Consider the boundary value problem

$$\begin{cases} -\Delta u = -au + bu^2 - du^3 - f(u) - \frac{c}{u^\alpha}, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \tag{1}$$

where $\alpha \in (a, b)$, a, b, d and c are positive constants, and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, Δ is the Laplacian operator, and $f : [0, \infty) \to \mathbb{R}$ is a continuous function. We make the following assumptions:

(H_1) $f : [0, \infty) \to \mathbb{R}$ is nondecreasing continuous functions such that $\lim_{s\to+\infty} f(s) = \infty$.

(H_2) $\lim_{s\to+\infty} \frac{f(s)}{s} = 0$.

Note that (1) is as an infinite semipositone problems ($\lim_{u\to 0} F(u) = -\infty$, where $F(u) := -au + bu^2 - du^3 - f(u) - (c/u^\alpha)$).

in [9] the authors have studied the case when $F(u) := g(u)(c/u)$ where g is nonnegative and nondecreasing and $\lim_{u\to\infty} g(u) = \infty$. The case $g(u) := au - f(u)$ has been studied in [8],

2 The main result

In this section, we shall establish our existence result via the method of sub - supersolution. A function ψ is said to be a subsolution of (1) if it is in $C^2(\Omega) \cap C(\bar{\Omega})$ such that $\psi = 0$ on $\partial\Omega$ and

$$-\Delta\psi \leq -a\psi + \psi^2 - d\psi^3 - f(\psi) - \frac{c}{\psi^\alpha} \quad in\ \Omega$$

and z is said supersolution of (1) if it is in $C^2(\Omega) \cap C(\bar{\Omega})$ such that $z = 0$ on $\partial\Omega$ and

$$-\Delta z \geq -az + z^2 - dz^3 - f(z) - \frac{c}{z^\alpha} \quad in\ \Omega$$

Then it is well known that if there exist a subsolution ψ and supersolution z such that $\psi \leq z$ in Ω then (1) has a solution u such that$\psi \leq u \leq z$, see [4].

***Theorem* 1.** *Let (H1) and (H2) hold, Then there exists positive constants* $b_0 := b_0(a, d, \Omega)$ *and* $c_0 := c_0(a, b, d, \Omega)$ *such that for* $b \geq b_0$ *and* $c \leq c_0$*, problem* (1) *has a positive solution.*

Proof. Let $\lambda_1 > 0$ be the first eigenvalue of the operator $-\Delta$ with Dirichlet boundary condition and ϕ_1 be the corresponding eigenfunction satisfying $\phi_1 > 0$ in Ω and $\frac{\partial\phi_1}{\partial v} < 0$ on $\partial\Omega$, where ν is outward normal vector on $\partial\Omega$ and $\| \phi_1 \|_\infty = 1$, see [5].
Note that λ_1 and ϕ_1 satisfy:

$$-\Delta\phi_1 = \lambda_1\phi_1 \quad in\ \Omega$$
$$\phi_1 = 0 \quad on\ \partial\Omega$$

Let $\sigma > 0$, $\mu > 0$, $m > 0$ be such that

$$\left(\frac{2}{1+\alpha}\right)\left\{\left(\frac{1-\alpha}{1+\alpha}\right)|\nabla\phi_1|^2 - \lambda_1\phi_1^2\right\} \geq m \quad in\ \bar{\Omega}_\delta, \tag{2}$$

and $\phi_1 \in [\mu, 1]$ in $\Omega\backslash\bar{\Omega}_\delta$ where $\bar{\Omega}_\delta := \{x \in \Omega :\ d(x, \partial\Omega) \leq \delta\}$. This is possible since $|\nabla\phi_1| \neq 0$ on $\partial\Omega$ while $\phi_1 = 0$ on $\partial\Omega$.
Let $b_0 > 2\sqrt{ab}$ and $P(s) = -as + bs^2 - ds^3$. Then the zeros of P(s) are $0,\ R_1 = \dfrac{b - \sqrt{b^2 - 4ad}}{2d}$ and $R_2 = \dfrac{b + \sqrt{b^2 - 4ad}}{2d}$. We note that P(s) can be factored as $P(s) = -ds(s - R_1)(s - R_2)$.
Let $r = \dfrac{b - \sqrt{b^2 - 3ad}}{3d}$ denote the first positive zero of $P'(s)$. since P(s) is convex on $(0, \frac{b}{3d})$ and $r < \frac{b}{3d}$, we have

$$\rho := -\inf_{s\in[0,R_2]} P(s) < a(b - \sqrt{b^2 - 3ad}/3d) = ar$$

(see 1) We note that

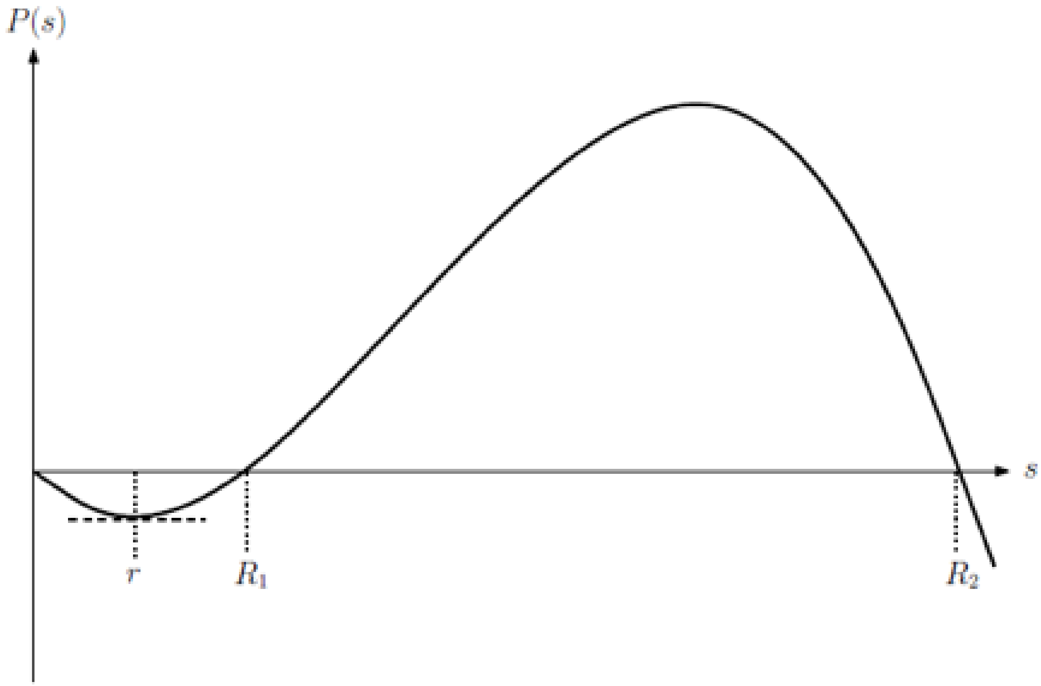

Figure 1: Graph of P(s).

$$\frac{\rho}{R_2} < \frac{a(b-\sqrt{b^2-3ad}/3d)}{b-\sqrt{b^2-4ad}/2d} = \frac{2a^2d}{(b-\sqrt{b^2-4ad})(b+\sqrt{b^2-3ad})} \to 0 \ as \ b \to \infty$$
$$\frac{R_2}{R_1} = \frac{b+\sqrt{b^2-4ad}}{b-\sqrt{b^2-4ad}} = \frac{(b+\sqrt{b^2-4ad})}{4ad} \to \infty \ as \ b \to \infty$$

Hence there exists $b_0^{(1)} := b_0^{(1)}(a,d,\Omega)$ such that for every $b > b_0^{(1)}$ we have

$$\frac{\rho}{R_2} < \frac{m}{6}, \tag{3}$$

$\left[\frac{R_2}{2}\mu^{\frac{2}{1+\alpha}}, \frac{R_2}{2}\right] \subset (R_1, R_2)$ and $k_\mu := \inf_{s\in\left[\frac{R_2}{2}\mu^{\frac{2}{1+\alpha}}, \frac{R_2}{2}\right]} P(s) > 0$. Next we see that

$$\begin{aligned}\frac{k_\mu}{R_2} &= \frac{\min\left\{P(\frac{R_2}{2}\mu^{\frac{2}{1+\alpha}}), P(\frac{R_2}{2})\right\}}{R_2} \\ &= \min\left\{d\frac{R_2}{2}\mu^{\frac{2}{1+\alpha}}\left(\frac{R_2}{2}\mu^{\frac{2}{1+\alpha}} - R_1\right)\left(1-\frac{\mu^{\frac{2}{1+\alpha}}}{2}\right), d\frac{R_2}{4}\left(\frac{R_2}{2} - R_1\right)\right\} \to \infty \ as \ b \to \infty\end{aligned}$$

and hence there exists $b_0^{(2)} := b_0^{(2)}(a,d,\Omega)$ such that foor every $b > b_0^{(2)}$ we have

$$\frac{k_\mu}{R_2} > \frac{2\lambda_l}{1+\alpha}.$$

Finally from (H1) and (H2), $f(R2) \to \infty$ and $f(R2/2)/(R2/2) \to 0$ as $b \to \infty$.
Thus there exists $b_0^{(3)} := b_0^{(3)}(a, d, \Omega)$ such that for every $b > b_0^{(3)}$ we have $f(R_2) \geq 0$ and

$$f\left(\frac{R_2}{2}\phi_1^{\frac{2}{1+\alpha}}\right) \leq f(\frac{R_2}{2}) \leq \min\{\lambda_1, \frac{m}{3}\}(\frac{R_2}{2}). \tag{4}$$

For a given $a, d > 0$, define $b_0 := \max\{b_0^{(1)}, b_0^{(2)}, b_0^{(3)}\}$ and

$$c_0 := c_0(a, b, d, \Omega) := \min\left\{\frac{m}{3}(\frac{R_2}{2})^{1+\alpha}, (\frac{R_2}{2})^{\alpha}\mu^{2\alpha/1+\alpha}(k_\mu - \frac{2\lambda_1}{1+\alpha}R_2)\right\},$$

and let $b \geq b_0$ and $c \leq c_0$. We will show that $\psi := R\phi_1^{2/1+\alpha}$ is a subsolution of (1), where $R := \frac{R_2}{2}$.
We first note that

$$\nabla\psi = R\left(\frac{2}{1+\alpha}\right)\phi_1^{\frac{1-\alpha}{1+\alpha}}\nabla\phi_1$$

and

$$\begin{aligned}
-\Delta\psi &= -R\left(\frac{2}{1+\alpha}\right)\left\{\phi_1^{\frac{1-\alpha}{1+\alpha}}\Delta\phi_1 + \left(\frac{1-\alpha}{1+\alpha}\right)\phi_1^{-\frac{2\alpha}{1+\alpha}}|\nabla\phi_1|^2\right\} \\
&= R\left(\frac{2}{1+\alpha}\right)\frac{1}{(\phi_1^{\frac{2}{1+\alpha}})^\alpha}\left\{\lambda_1\phi_1^2 - (\frac{1-\alpha}{1+\alpha})|\nabla\phi_1|^2\right\}.
\end{aligned}$$

Next for $x \in \bar{\Omega}_\delta$ since $\frac{1}{(\phi_1^{\frac{2}{1+\alpha}})^\alpha} \geq 1$, from (2), (3), (4) and $c \leq c_0$ we have

$$\begin{aligned}
-\Delta\psi &= R\left(\frac{2}{1+\alpha}\right)\frac{1}{(\phi_1^{\frac{2}{1+\alpha}})^\alpha}\left\{\lambda_1\phi_1^2 - \left(\frac{1-\alpha}{1+\alpha}\right)|\nabla\phi_1|^2\right\} \\
&\leq -mR\frac{1}{(\phi_1^{\frac{2}{1+\alpha}})^\alpha} \\
&= -\frac{mR}{3\left((\phi_1^{\frac{2}{1+\alpha}})\right)^\alpha} - \frac{mR}{3\left((\phi_1^{\frac{2}{1+\alpha}})\right)^\alpha} - \frac{mR}{3\left((\phi_1^{\frac{2}{1+\alpha}})\right)^\alpha} \\
&\leq -\frac{mR}{3} - \frac{mR}{3} - \frac{mR}{3\left((\phi_1^{\frac{2}{1+\alpha}})\right)^\alpha} \\
&\leq -\rho - f\left(R\phi_1^{\frac{2}{1+\alpha}}\right) - \frac{mR^{1+\alpha}/3}{\left(R\phi_1^{\frac{2}{1+\alpha}}\right)^\alpha} \\
&\leq -a\psi + b\psi^2 - d\psi^3 - f(\psi) - \frac{c}{\psi^\alpha}.
\end{aligned}$$

Also for $x \in \Omega\backslash\bar{\Omega}_\delta$, since $0 < \mu \leq \phi$, from (4) and $c \leq c_0$,

$$
\begin{aligned}
-\Delta\psi &= R\left(\frac{2}{1+\alpha}\right)\frac{1}{\left(R\phi_1^{\frac{2}{1+\alpha}}\right)^{\alpha}}\left\{\lambda_1\phi_1^2-\left(\frac{1-\alpha}{1+\alpha}\right)|\nabla\phi_1|^2\right\}\\
&\le R\left(\frac{2}{1+\alpha}\right)\lambda_1\phi^{\frac{2}{1+\alpha}}\\
&\le R\left(\frac{2}{1+\alpha}\right)\lambda_1\\
&= 2\left[R\left(\frac{2}{1+\alpha}\right)\lambda_1\right]-R\left(\frac{2}{1+\alpha}\right)\lambda_1\\
&\le \frac{4\lambda_1}{1+\alpha}R-R\lambda_1\\
&\le k_\mu-\frac{c}{\left(R\mu^{\frac{2}{1+\alpha}}\right)^{\alpha}}-f\left(R\phi_1^{\frac{2}{1+\alpha}}\right)\\
&\le -a\psi+b\psi^2-d\psi^3-f(\psi)-\frac{c}{\psi^\alpha}. \qquad (5)
\end{aligned}
$$

According to (**??**) and (**??**), we can conclude that ψ is a subsolution of (1). We also show that $z := R_2$ is a supersolution, by noting that

$$
-\Delta z = 0 \ge -f(z)-\frac{c}{z^\alpha} = -az+bz^2-dz^3-f(z)-\frac{c}{z^\alpha}.
$$

Further $z \ge \psi$. Thus, (1) has a positive solution. This completes the proof of Theorem 2.1.

□

3 Extension of (1) to system (6)

In this section, we consider the extension of (1) to the following system:

$$
\begin{cases}
-\Delta u = -a_1u+b_1u^2-d_1u^3-f_1(u)-\frac{c_1}{v^\alpha}, & x\in\Omega,\\
-\Delta u = -a_2u+b_2u^2-d_2u^3-f_2(u)-\frac{c_2}{v^\alpha}, & x\in\Omega,\\
u=0=v, & x\in\partial\Omega,
\end{cases} \qquad (6)
$$

where $\alpha\in(0,1)$, a_1, a_2, b_1, b_2, d_1, d_2, c_1 and c_2 are positive constants, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and $f_i:[0,\infty)\to\mathbb{R}$ is a continuous function for $i=1,2$. We make the following assumptions

(H_1) $f_i:[0,+\infty)\to\mathbb{R}$ is nondecreasing continuous functions such that $\lim_{s\to+\infty} f_i(s)=\infty$ for $i=1,2$.

(H_2) $\lim_{s\to+\infty} \frac{f_1(s)}{s} = 0$ for $i = 1, 2$.

We prove the following result by finding sub-super solutions to infinite semipositone system (6).

Theorem 2. *Let (H3) and (H4) hold, Then there exists positive constants* $b_0^* := b_0^*(a_1, a_2, d_1, d_2, \Omega)$ *and* $c_0^* := b_0^*(a_1, a_2, b_1, b_2, d_1, d_2, \Omega)$ *such that for* $\min\{b_1, b_2\} \geq b_0^*$ *and* $\max\{c_1, c_2\} \leq c_0^*$*, problem* (6) *has a positive solution.*

Proof. Let $(R_1^{(i)}, R_2^{(i)}, \rho^{(i)}, k_\mu^{(i)})$, $P_i(s) := -a_i s + b_i s^2 - d_i s^3$ for $i = 1, 2$ be given, as in section 2. By the same argument as in section 2, there exists $b_0^* := b_0^*(a_1, a_2, d_1, d_2, \Omega)$ such that for $\min\{b_1, b_2\} > b_0^*$ we have

$$\frac{\rho^{(i)}}{R_2^{(i)}} < \frac{m}{6}, \quad \frac{k_\mu^{(i)}}{R_2^{(i)}} > \frac{2\lambda}{1+\alpha},$$

and $f_i\left(\frac{R_2^{(i)}}{2}\phi_1^{\frac{2}{1+\alpha}}\right) \leq \min\{\lambda_1, \frac{m}{3}\}\left(\frac{R_2^{(i)}}{2}\right)$ for $i = 1, 2$. Define

$$\begin{aligned}
c_0^* &:= c_0^*(a_1, a_2, b_1, b_2, d_1, d_2, \Omega) \\
&:= \min\Bigg\{\frac{m}{3}\left(\frac{R_2^{(1)}}{2}\right)\left(\frac{R_2^{(2)}}{2}\right)^\alpha, \frac{m}{3}\left(\frac{R_2^{(1)}}{2}\right)^\alpha\left(\frac{R_2^{(2)}}{2}\right), \left(\frac{R_2^{(2)}}{2}\right)^\alpha \mu^{2\alpha/1+\alpha}\left(k_\mu^{(1)} - \frac{2\lambda_1}{1+\alpha}R_2^{(1)}\right), \\
&\left(\frac{R_2^{(1)}}{2}\right)^\alpha \mu^{2\alpha/1+\alpha}\left(k_\mu^{(2)} - \frac{2\lambda_1}{1+\alpha}R_2^{(2)}\right)\Bigg\}
\end{aligned}$$

and $(\psi_1\psi_2) := (R^{(1)}\phi_1^{2/1+\alpha}, R^{(2)}\phi_2^{2/1+\alpha})$, where $R^{(i)} = R_2^{(i)}/2$. Let $\min\{b_1, b_2\} > b_0^*$ and $\max\{c_1, c_2\} \leq c_0^*$, then for $x \in \bar{\Omega}_\delta$ we have

$$\begin{aligned}
-\Delta\psi_1 &= R^{(1)}\left(\frac{2}{1+\alpha}\right)\frac{1}{\left(\phi_1^{\frac{2}{1+\alpha}}\right)^\alpha}\left\{\lambda_1\phi_1^2 - \left(\frac{1-\alpha}{1+\alpha}\right)|\nabla\phi_1|^2\right\} \\
&\leq -mR^{(1)}\frac{1}{\left(\phi_1^{\frac{2}{1+\alpha}}\right)^\alpha} \\
&\leq -\frac{mR^{(1)}}{3} - \frac{mR^{(1)}}{3} - \frac{mR^{(1)}}{3\left(\phi_1^{\frac{2}{1+\alpha}}\right)^\alpha} \\
&\leq -\rho^{(1)} - f(R^{(1)}\phi_1^{\frac{2}{1+\alpha}}) - \frac{mR^{(1)}[R^{(2)}]^\alpha/3}{(R^{(2)}\phi_1^{\frac{2}{1+\alpha}})^\alpha} \\
&\leq -a\psi_1 + b\psi_1^2 - d\psi_1^3 - f(\psi_1) - \frac{c_1}{\psi_2^\alpha}.
\end{aligned}$$

And for $x \in \Omega\backslash\bar{\Omega}_\delta$, we have

$$
\begin{aligned}
-\Delta\psi_1 &= R^{(1)}\left(\frac{2}{1+\alpha}\right)\frac{1}{\left(\phi_1^{\frac{2}{1+\alpha}}\right)^{\alpha}}\left\{\lambda_1\phi_1^2-\left(\frac{1-\alpha}{1+\alpha}\right)|\nabla\phi_1|^2\right\}\\
&\le R^{(1)}\left(\frac{2}{1+\alpha}\right)\lambda_1\\
&= 2\left[R^{(1)}\left(\frac{2}{1+\alpha}\right)\lambda_1\right]-R^{(1)}\left(\frac{2}{1+\alpha}\right)\lambda_1\\
&\le \frac{4\lambda_1}{1+\alpha}R^{(1)}-R^{(1)}\lambda_1\\
&\le k_\mu^{(1)}-\frac{c_2}{R^{(2)}\mu^{\frac{2}{1+\alpha}}}-f\left(R^{(1)}\phi_1^{\frac{2}{1+\alpha}}\right)\\
&\le -a_1\psi_1+b_1\psi_1^2-d_1\psi_1^3-f(\psi_1)-\frac{c_1}{\psi_2^{\alpha}}.
\end{aligned}
$$

Similary

$$
-\Delta\psi_2 \le -a_2\psi_2+b_2\psi_2^2-d_2\psi_2^3-f(\psi_2)-\frac{c_2}{\psi_1^{\alpha}},\quad x\in\Omega
$$

Thus the (ψ_1,ψ_2) is a subsolution of (6). It is obvious that $(z_1,z_2):=(R_2^{(1)},R_2^{(2)})$ is a supersolution of (6), such that $(z_1,z_2)\ge(\psi_1,\psi_2)$. Thus Theorem 3.1 is proven.

□

4 Extension of (1) to problem (7)

In this section, we consider the extension of (1.1) to the following problem:

$$
\begin{cases}
-\Delta_p u = -au+bu^2-du^3-f(u)-\frac{c}{u^{\alpha}}, & x\in\Omega,\\
u=0, & x\in\partial\Omega,
\end{cases}
\tag{7}
$$

where $\Delta_{p^z}=div(|\nabla z|^{p-2}\nabla z)$, $p>1$, $\alpha\in(0,1)$, a, b, d and c are positive constants, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and $f:[0,\infty)\to\mathbb{R}$ is a continuous function. Then we have the following result.

***Theorem* 3.** *Let (H1) and (H2) hold, Then there exists positive constants $b_0^{**}:=b_0^{**}(a,d,\Omega)$ and $c_0^{**}:=c_0^{**}(a,d,\Omega)$ such that for $b\ge b_0^{**}$ and $c\le c_0^{**}$, problem (7) has a positive solution.*

Proof. We shall establish Theorem 4.1 by constructing positive sub-super solutions to equation (7). Let λ_1 be the first eigenvalue of the problem

$$
-\Delta_p\phi_1=\lambda_1\phi_1^{p-1},\quad x\in\Omega,\quad \phi_1=0,\quad x\in\partial\Omega,
$$

where ϕ_1 denote the corresponding eigenfunction, satisfying $\phi_1 > 0$ in Ω and $|\nabla \phi_1| > 0$ on $\partial\Omega$, see [5]. Without loss of generality, we let $\| \phi_1 \|_\infty = 1$. Let $\delta > 0$, $\mu > 0$, $m > 0$ be such that

$$\left(\frac{p}{p-1+\alpha}\right)^{p-1} \left\{\frac{(1-\alpha)(p-1)}{p-1+\alpha}|\nabla \phi_1|^p - \lambda_1 \phi_1^p\right\} \geq m \quad in\ \bar{\Omega}_\delta,$$

and $\phi_1 \in [\mu, 1]$ in $\Omega \backslash \bar{\Omega}_\delta$, where $\bar{\Omega}_\delta := \{x \in \Omega :\ d(x, \partial\Omega) \leq \delta\}$. This is possible since $|\nabla \phi_1| \neq 0$ on $\partial\Omega$ while $\phi_1 = 0$ on $\partial\Omega$.

Also let R_1, R_2 be as in section 2 and b_0^{**} be such that for every $b > b_0^{**}$

$$\frac{\rho}{R_2^{p-1}} < \frac{m}{6}, \quad \frac{k_\mu}{R_2^{p-1}} > \left(\frac{\lambda_1}{2}\right)\left(\frac{p}{p-1-\alpha}\right)^{p-1},$$

and

$$f\left(\left[\frac{R_2}{2}\right]^{p-1} \phi_1^{\frac{p}{p-1+\alpha}}\right) \leq \min\left\{\lambda_1, \frac{m}{3}\right\}\left(\frac{R_2}{2}\right)^{p-1}.$$

Define

$$\begin{aligned} c_0^{**} &:= c_0^{**}(a, b, d, \Omega) \\ &:= \min\left\{\frac{m}{3}\left(\frac{R_2}{2}\right)^{(p-1)(1+\alpha)}, \left(\frac{R_2}{2}\right)^{\alpha(1+\alpha)} \mu^{\frac{\alpha p}{p-1+\alpha}}\left[k_\mu - R_2\lambda_1\left(\frac{p}{p-1+\alpha}\right)^{p-1}\right]\right\} \end{aligned}$$

and $\psi := R\phi_1^{\frac{p}{p-1+\alpha}}$. Then

$$\nabla\psi = R\left(\frac{p}{p-1+\alpha}\right)\phi_1^{\frac{1-\alpha}{p-1+\alpha}}\nabla\phi_1$$

and

$$\begin{aligned} \Delta_p\psi &= div(|\nabla\psi|^{p-2}\nabla\psi) \\ &= R^{p-1}\left(\frac{p}{p-1+\alpha}\right)^{p-1} div\left(\phi_1^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}}|\nabla\phi_1|^{p-2}\nabla\phi_1\right) \\ &= R^{p-1}\left(\frac{p}{p-1+\alpha}\right)^{p-1}\left\{\nabla\left(\phi_1^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}}\right)|\nabla\phi_1|^{p-2}\nabla\phi_1 + \phi_1^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}}\Delta_p\phi_1\right\} \\ &= R^{p-1}\left(\frac{p}{p-1+\alpha}\right)^{p-1}\left\{\frac{(1-\alpha)(p-1)}{p-1+\alpha}\phi_1^{\frac{-\alpha p}{p-1+\alpha}}|\nabla\phi_1|^p - \lambda_1\phi_1^{\frac{p(p-1)}{p-1+\alpha}}\right\} \\ &= R^{p-1}\left(\frac{p}{p-1+\alpha}\right)^{p-1}\frac{1}{\phi_1^{\frac{p}{p-1+\alpha}}}\left\{\frac{(1-\alpha)(p-1)}{p-1+\alpha}|\nabla\phi_1|^p - \lambda_1\phi_1^p\right\}, \end{aligned}$$

thus,

$$-\Delta_p\psi = R^{p-1}\left(\frac{p}{p-1+\alpha}\right)^{p-1}\frac{1}{\left(\phi_1^{\frac{p}{p-1+\alpha}}\right)}\left\{\lambda_1\phi_1^p - \frac{(1-\alpha)(p-1)}{p-1+\alpha}|\nabla\phi_1|^p\right\}.$$

By the same argument as in the proof of theorem 2.1, we can show that ψ is a subsolution of (7) for $b \geq b_0^{**}$ and $c \leq c_0^{**}$. Next, it is easy to check that $z := R_2$ is a supersolution of (7) with $z \geq \psi$. Hence (7) has a positive solution and the proof is complete.

□

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